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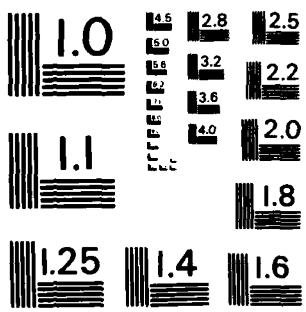

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EXISTENCE OF VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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ABSTRACT

Equations of Hamilton-Jacobi type arise in many areas of application, including the calculus of variations, control theory and differential games. Recently M. G. Crandall and P. L. Lions introduced the class of "viscosity" solutions of these equations and proved uniqueness within this class. This paper discusses the existence of these solutions under assumptions closely related to the ones which guarantee the uniqueness.

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SIGNIFICANCE AND EXPLANATION

Equations of Hamilton-Jacobi type arise in many areas of application, including the calculus of variations, control theory and differential games. However, nonlinear first order partial differential equations almost never have global classical solutions, and one must deal with generalized solutions. The correct class of generalized solutions for equations of Hamilton-Jacobi type has recently been established by M. G. Crandall, L. C. Evans and P. L. Lions. Here we give some existence results concerning this solution, under assumptions similar to the ones guaranteeing its uniqueness.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EXISTENCE OF VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

Panagiotis E. Souganidis

INTRODUCTION

Recently M. G. Crandall and P. L. Lions ([2]) introduced the notion of viscosity solution for nonlinear scalar partial differential equations of the form

$$F(y,u(y),Du(y)) = 0 \quad \text{for} \quad y \in \mathcal{O} \tag{0.1}$$

where $\mathcal{O}$ is an open set in $\mathbf{R}^m$, $F : \mathcal{O} \times \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}$ is continuous and $Du = (\partial u/\partial y_1, \ldots, \partial u/\partial y_m)$ denotes the gradient of u (also see M. G. Crandall, P. L. Lions and L. C. Evans [1]). They used this notion to prove uniqueness and stability for a wide class of equations of the form (0.1), in particular for the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(t,x,u,Du) = 0 & \text{in} \quad \mathbf{R}^N \times (0,T] \\ u(x,0) = u_0(x) & \text{in} \quad \mathbf{R}^N \end{cases} \tag{0.2}$$

and the stationary problem

$$u + \lambda H(x,u,Du) = n(x) \quad \text{in} \quad \mathbf{R}^N . \tag{0.3}$$

Moreover they proved existence of the viscosity solution of the model problems

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(Du) = 0 & \text{in} \quad \mathbf{R}^N \times (0,T] \\ u(x,0) = u_0(x) & \text{in} \quad \mathbf{R}^N \end{cases} \tag{0.4}$$

and

$$u + H(Du) = n(x) \quad \text{in} \quad \mathbf{R}^N . \tag{0.5}$$

This paper discusses the existence of the viscosity solution of the more general problems (0.2) and (0.3). The assumptions made here are closely related to the ones for which M. G. Crandall and P. L. Lions proved the uniqueness of this solution.

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We now formulate typical hypotheses and describe the results. As far as $H : [0,T] \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is concerned throughout this discussion we will assume

$$(H1)\begin{cases} H \in C([0,T] \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N) \text{ is uniformly continuous in} \\ [0,T] \times \mathbf{R}^N \times [-R,R] \times B_N(0,R) \quad \text{for each} \quad R > 0 \end{cases} \qquad (*)$$

and

$$(H2)\begin{cases} \text{There is a constant } C > 0 \text{ so that} \\ \qquad C = \sup_{Q_T} |H(t,x,0,0)| < \infty \ . \end{cases} \qquad (**)$$

Moreover we require some monotonicity of H with respect to u. More precisely we assume

$$(H3)\begin{cases} \text{For } R > 0 \text{ there is a } \gamma_R \in \mathbf{R} \text{ such that} \\ H(t,x,r,p) - H(t,x,s,p) \geq \gamma_R(r-s) \quad \text{for} \quad x \in \mathbf{R}^N, -R \leq s \leq r \leq R \\ 0 \leq t \leq T \quad \text{and} \quad p \in \mathbf{R}^N \ . \end{cases}$$

Finally we will have to restrict the nature of the joint continuity of H. The following two assumptions will be used:

(*) $C^k(\mathcal{O})$ is the space of k times continuously differentiable functions defined on $\mathcal{O}$.

$C_b^k(\mathcal{O})$ consists of functions in $C^k(\mathcal{O})$ which together with their derivatives are bounded

$C_0^k(\mathcal{O})$ consists of functions in $C^k(\mathcal{O})$ which togeher with their derivatives have compact support

$B_N(x_0,R) = \{x \in \mathbf{R}^N : |x-x_0| \leq R\}$.

(**) $Q_T = \mathbf{R}^N \times (0,T]$, $\bar{Q}_T = \mathbf{R}^N \times [0,T]$ where $T \in (0,\infty)$

$Q_\infty = \mathbf{R}^N \times (0,\infty)$, $\bar{Q}_\infty = \mathbf{R}^N \times [0,\infty)$

$$\text{(H4)}\quad \begin{cases} \text{If } \Lambda_R(\alpha) = \sup\{|H(t,x,r,p) - H(t,y,r,p)| : |x-y| \le \alpha, |x-y||p| \le R, \\ \qquad\qquad\qquad\qquad\qquad |r| \le R,\ t \in [0,T]\} \\ \text{then } \lim_{\alpha\downarrow 0} \Lambda_R(\alpha) = 0 \text{ for any } R > 0 \end{cases}$$

$$\text{(H5)}\quad \begin{cases} \text{For } R > 0 \text{ there is a constant } C_R > 0 \text{ such that} \\ |H(t,x,r,p) - H(t,y,r,p)| \le C_R(1+|p|)|x-y| \quad \text{for } t \in [0,T],\ |r| \le R \\ \text{and } x,y,p \in \mathbf{R}^N . \end{cases}$$

The theorems are:

<u>Theorem 1</u>. Let $H : [0,T] \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ satisfy (H1), (H2), (H3) and either (H4) or (H5). For any $u_0 \in BUC(\mathbf{R}^N)$ there is a $T = T(\|u_0\|) > 0$ and $u \in BUC(\overline{Q}_T)$ such that u is the unique viscosity solution of (0.2) in $\tilde{Q}_T$.(*) If moreover γ_R in (H2) is independent of R, then (0.2) has a unique viscosity solution in $\overline{Q}_T$ for every $T > 0$.

<u>Theorem 2</u>. Let $H : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ satisfy (H1), (H2), (H3) and either (H4) or (H5). For any $n \in BUC(\mathbf{R}^N)$, there is a $\lambda_0 = \lambda(\|n\|, \gamma_R)$ such that for every λ, $0 \le \lambda < \lambda_0$, (0.3) has a unique viscosity solution $u \in BUC(\mathbf{R}^N)$

Several existence results for the problems (0.2) and (0.3) (including versions with boundary conditions) can be found in P. L. Lions [7,8]. His assumptions generalize (H5) but not (H4). However, for (0.2) he requires a Lipschitz condition in t. Moreover, W. H. Fleming ([4]) and A. Friedman ([6]) established earlier some existence results concerning (0.2) in the almost

(*) $BUC(\mathcal{O})$ is the space of bounded uniformly continuous functions defined on $\mathcal{O}$. If $u : \mathcal{O} \to \mathbf{R}$ then $\|u\| = \sup_{x\in\mathcal{O}} |u(x)|$

everywhere sense, under Lipschitz type assumptions for all the arguments of H and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$. (*) Finally, the scope of the existence results has been recently extended by G. Barles ([0]).

The paper is organized as follows. Section 1 recalls the definition and some basic properties of the viscosity solution of (0.2). It also contains some new results about this solution. Section 2 is devoted to the proof of theorem 1. Moreover, as an intermediate step towards the proof of this theorem, we give a result about the convergence of the viscosity approximations with certain explicit estimates. Sections 3 and 4 are devoted to the stationary problem and have the same structure as sections 1 and 2.

Finally, we would like to thank Professor M. G. Crandall for helpful discussions and good advice.

(*) $C_{(b)}^{0,1}(\mathcal{O})$ is the set of (bounded) Lipschitz continuous functions defined on $\mathcal{O}$.

Section 1

We begin this section with the definition of the viscosity solution of (0.2). We have

Definition 1.1 (5.1 [2]). Let $H \in C([0,T] \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N)$. A function $u \in C(Q_T)$ is a viscosity solution of

$$\frac{\partial u}{\partial t} + H(t,x,u,Du) = 0$$

if for every $\phi \in C^\infty(Q_T)$

(1.1) if $u - \phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then

$$\frac{\partial \phi}{\partial t}(x_0,t_0) + H(t_0,x_0,u(x_0,t_0), D\phi(x_0,t_0)) \leq 0$$

and

(1.2) if $u - \phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then

$$\frac{\partial \phi}{\partial t}(x_0,t_0) + H(t_0,x_0,u(x_0,t_0), D\phi(x_0,t_0)) \geq 0 \quad .$$

If moreover $u \in C(\bar{Q}_T)$ and $u(x,0) = u_0(x)$ in $\mathbf{R}^N$, we say that u is a viscosity solution of (0.2).

Remark 1.1. In a similar way $u \in C(\bar{Q}_T)$ is said to be a viscosity subsolution (respectively supersolution) of (0.3) if (1.1) (respectively (1.2)) holds and $u(x,0) \leq u_0(x)$ (respectively $u(x,0) \geq u_0(x)$) in $\mathbf{R}^N$.

Remark 1.2. Definition 1.1 and Remark 1.1 are a combination of Definition 2 and Lemma 4.1 of [1].

Next we state the theorem about the uniqueness of the viscosity solution of (0.2) as well as some other important results of [2] concerning this solution.

Theorem 1.1 (V.2[2]). Let $u,v \in PUC(\mathbf{R}^N)$ be viscosity solutions of the problems

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(t,x,u,Du) = 0 & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial v}{\partial t} + H(t,x,v,Dv) = 0 & \text{in } Q_T \\ v(x,0) = v_0(x) & \text{in } \mathbf{R}^N \end{cases}$$

respectively where $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies (H1), (H3) and either (H4) or (H5). Let $R_0 = \max(\|u\|, \|v\|)$ and $\gamma = \gamma_{R_0}$. Then for $t \in [0,T]$

(1.3) $$\|u(\cdot,t) - v(\cdot,t)\| \le e^{-\gamma t}\|u_0 - v_0\| \quad .$$

In particular (0.2) has at most one viscosity solution.

Proposition 1.1 (I.11 [2]). Let $T > 0$, $\gamma \in \mathbb{R}$ and $g,h \in C([0,T])$. Suppose that for every $\eta \in C^\infty((0,T))$, if $g-\eta$ attains a strict local maximum at $t_0 \in (0,T)$, we have

$$\eta'(t_0) + \gamma g(t_0) \le h(t_0) \quad .$$

Then for $0 \le s \le t \le T$

(1.4) $$e^{\gamma t}g(t) \le e^{\gamma s}g(s) + \int_s^t e^{\gamma\tau}h(\tau)d\tau \quad .$$

Remark 1.3. The assumptions on g in the above proposition are equivalent to saying that g is a viscosity solution of

$$g' + \gamma g \le h$$

as it is explained in [2].

Proposition 1.2 (VI.1[2]). For $\varepsilon > 0$ let $u_\varepsilon \in C_b(\overline{Q}_T)$ be a solution of

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \varepsilon\Delta u_\varepsilon + H_\varepsilon(t,x,u_\varepsilon,Du_\varepsilon) = 0 & \text{in } Q_T \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x) & \text{in } \mathbb{R}^N \end{cases}$$

with $\dfrac{\partial u_\varepsilon}{\partial t}, \dfrac{\partial u_\varepsilon}{\partial x_i \partial x_j} \in C(Q_T)$. Assume $H_\varepsilon \to H$ uniformly on $[0,T] \times \mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$. If $\varepsilon_n \downarrow 0$ and $u_{\varepsilon_n} \to u$ locally uniformly in Q_T, then $u \in C(Q_T)$ is a viscosity solution of

$$\frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 \quad \text{in } Q_T \quad .$$

If moreover $u_{0\varepsilon} \to u_0$ uniformly in $\mathbb{R}^N$ and $u_{\varepsilon_n} \to u$ uniformly in $\overline{Q}_T$, then u is a viscosity solution of (0.2).

Proposition 1.3 (I.2[2]). Let $u_n \in C(\bar{Q}_T)$ be a viscosity solution of

$$\begin{cases} \dfrac{\partial u_n}{\partial t} + H_n(t,x,u_n,Du_n) = 0 & \text{in } Q_T \\ u_n(x,0) = u_{0n}(x) & \text{in } R^N . \end{cases}$$

Assume $H_n \to H$ uniformly on $[0,T] \times \mathbf{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$. If $u_n \to u$ locally uniformly in Q_T, then u is a viscosity solution of

$$\frac{\partial u}{\partial t} + H(t,x,u,Du) = 0 \quad \text{in } Q_T .$$

If moreover $u_{0n} \to u_0$ uniformly on R^N and $u_n \to u$ uniformly on $\bar{Q}_T$, then u is a viscosity solution of (0.2).

Now we give a result which describes the evolution in time of the "off the diagonal" difference of the viscosity solutions of two problems of the form (0.2). To this end choose $\beta \in C_0^\infty(\mathbf{R}^N)$ and $\gamma \in C_0^\infty(\mathbf{R})$ so that

$$\text{(1.5)} \qquad \begin{cases} 0 \le \beta \le 1,\ \beta(0) = 1,\ |D\beta| \le 2 \text{ and} \\ \beta(x) = 0 \text{ if } |x| > 1 \end{cases}$$

and

$$\text{(1.6)} \qquad \begin{cases} 0 \le \gamma \le 1,\ \gamma(0) = 1 \text{ and} \\ \gamma(t) = 0 \text{ if } |t| > 1 . \end{cases}$$

For $\varepsilon > 0$ set $\beta_\varepsilon(x) = \beta(\frac{x}{\varepsilon})$ and $\gamma_\varepsilon(t) = \gamma(\frac{t}{\varepsilon})$. We have

Proposition 1.4. Let $u, \bar{u} \in BUC(\bar{Q}_T)$ be viscosity solutions of the problems

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(t,x,u,Du) = 0 & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \bar{H}(t,x,\bar{u},D\bar{u}) = 0 & \text{in } Q_T \\ u(\bar{x},0) = \bar{u}_0(x) & \text{in } \mathbf{R}^N \end{cases}$$

respectively, where $u_0, \bar{u}_0 \in BUC(\mathbf{R}^N)$ and $H, \bar{H} : [0,T] \times \mathbf{R}^N \times R \times \mathbf{R}^N \to R$ satisfy (H1) and (H3) with the same constant $\gamma_R \le 0$ for each $R > 0$. Let $R_0 = \max(\|u\|,\|\bar{u}\|)$ and $\gamma = \gamma_{R_0}$. If for $R > R_0$ and $\varepsilon > 0$, D_ε, A_ε are such that

$$D_\varepsilon = \{(x,y) \in \mathbf{R}^N \times \mathbf{R}^N : |x-y| < \varepsilon\}$$

and

$$A_\varepsilon = \{(t,x,y,r,p) \in [0,T] \times R^N \times R^N \times R \times R^N : (x,y) \in D_\varepsilon,$$

$$|r| < \min(\|u\|,\|v\|),\ |p| < \min\left(\frac{6Re^{|\gamma|T}}{\varepsilon} + 1,\ L\right)\}$$

where

$$L = \min\left(\sup_{[0,T]} \|Du(\cdot,\tau)\|,\ \sup_{[0,T]} \|D\bar{u}(\cdot,\tau)\|\right) \quad (*)$$

then for every $\tau \in [0,T]$

$$\sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau)-\bar{u}(y,\tau)| + 3Re^{-\gamma\tau}\beta_\varepsilon(x-y)\} < e^{-\gamma\tau} \sup_{(x,y)\in D_\varepsilon} \{|u_0(x)-\bar{u}_0(y)| +$$

$$+ 3R\beta_\varepsilon(x-y)\} + e^{-\gamma\tau}\tau \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p) - \bar{H}(t,y,r,p)| \quad . \tag{1.7}$$

<u>Remark 1.4</u>. The assumption that H, $\bar{H}$ satisfy (H3) with the same constant is not important. It is made only for simplicity. Moreover one can always reduce to the case $\gamma_R < 0$ for every $R > 0$.

<u>Proof of proposition 1.4</u>. For $\tau \in [0,T]$ let $m^\pm(\tau)$ be defined by

$$m^\pm(\tau) = \sup_{(x,y)\in D_\varepsilon} \{(u(x,\tau) - \bar{u}(y,\tau))^+ + 3Re^{-\gamma\tau}\beta_\varepsilon(x-y)\} \quad .^{(**)} \tag{1.8}$$

Then obviously (1.7) follows from

$$m^\pm(\tau) < e^{-\gamma\tau}m^\pm(0) + e^{-\gamma\tau} \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p) - \bar{H}(t,y,r,p)| \int_0^\tau e^{\gamma\sigma}d\sigma \quad . \tag{1.9}$$

(*) For $u : \mathcal{O} \to \mathbf{R}$, $\|Du\|$ denotes its Lipschitz constant. If u is not Lipschitz continuous, then $\|Du\| = \infty$.

(**) $r^+(r^-)$ denotes the maximum of r (respectively $-r$) and 0.

Moreover, since $m^{\pm} \in C([0,T])$ ($u,v \in BUC(\bar{\Omega}_T)$), in view of Proposition 1.1 and Remark 1.3, it suffices to show that $m^{\pm}(\tau)$ is a viscosity solution in $(0,T)$ of

$$(1.10) \qquad (m^{\pm})' + \gamma m^{\pm} \le \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p) - \bar{H}(t,y,r,p)| \ .$$

Finally here we work only with m^+, since for the proof of the m^- case, one uses exactly the same arguments.

To this end, for $n \in C^\infty((0,T))$ let $\hat{\tau} \in (0,T)$ be such that $m^+ - n$ attains a strict maximum on $I = [\hat{\tau}-\alpha, \hat{\tau}+\alpha] \subset (0,T)$ for some $\alpha > 0$. We want to show that

$$(1.11) \qquad n'(\hat{\tau}) + \gamma m^+(\hat{\tau}) \le \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p) - \bar{H}(t,y,r,p)| \ .$$

If $m^+(\hat{\tau}) = 3Re^{-\gamma\hat{\tau}}$ then $\hat{\tau}$ is a local maximum of $3Re^{-\gamma\tau} - n(\tau)$ in $(0,T)$, since for every $\tau \in [0,T]$ it is

$$m^+(\tau) \ge 3Re^{-\gamma\tau} \ .$$

Thus

$$n'(\hat{\tau}) = -\gamma 3Re^{-\gamma\hat{\tau}} = -\gamma m^+(\hat{\tau})$$

and (1.11) is obviously satisfied. So without any loss of generality we may assume that

$$(1.12) \qquad m^+(\hat{\tau}) > 3Re^{-\gamma\hat{\tau}} \ .$$

In this case and for $\delta > 0$, let $\Phi : \mathbf{R}^N \times \mathbf{R}^N \times I \times I \to \mathbf{R}$ be defined by

$$(1.13) \qquad \begin{aligned} \Phi(x,y,\tau,s) = (u(x,\tau) - \bar{u}(y,s))^+ + 3Re^{-\gamma\frac{(\tau+s)}{2}}\beta_\varepsilon(x-y) \ + \\ + (3R+2\|n\|)\gamma_\delta(\tau-s) - n(\tfrac{\tau+s}{2}) \end{aligned}$$

where $\gamma_\delta(t) = \gamma(t/\delta)$ is defined by (1.6). Since Φ is bounded on $\mathbf{R}^N \times \mathbf{R}^N \times I \times I$, for every $\delta > 0$ there is a point $(x_1, y_1, \tau_1, s_1) \in \mathbf{R}^N \times \mathbf{R}^N \times I \times I$ such that

$$\Phi(x_1,y_1,\tau_1,s_1) > \sup_{R^N\times R^N\times I\times I} \Phi - \delta \ .$$

Next select $\zeta \in C_0^\infty(R^N \times R^N)$ satisfying $0 \le \zeta \le 1$, $\zeta(x_1,y_1) = 1$, $|D\zeta| \le 1$ and define $\Psi : R^N \times R^N \times I \times I \to R$ by

$$\Psi(x,y,\tau,s) = \Phi(x,y,\tau,s) + 2\delta\zeta(x,y) \ . \tag{1.14}$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1,\tau_1,s_1) = \Phi(x_1,y_1,\tau_1,s_1) + 2\delta > \sup_{R^N\times R^N\times I\times I} \Phi + \delta$$

there exists a $(x_0,y_0,\tau_0,s_0) \in R^N \times R^N \times I \times I$ such that

$$\Psi(x_0,y_0,\tau_0,s_0) \ge \Psi(x,y,\tau,s) \quad \text{for every} \quad (x,y,\tau,s) \in R^N \times R^N \times I \times I \ . \tag{1.15}$$

Moreover for $\delta < R/2$

$$|\tau_0-s_0| \le \delta \ . \tag{1.16}$$

Indeed suppose not. Then (1.15) and (1.6) imply

$$2R + 3Re^{-\gamma(\hat{\tau}+\alpha)} - n(\frac{\tau_0+s_0}{2}) + 2\delta \ge \Psi(x_0,y_0,\tau_0,s_0) \ge \Psi(x,x,\hat{\tau}+\alpha,\hat{\tau}+\alpha)$$

$$\ge 3Re^{-\gamma(\hat{\tau}+\alpha)} + 3R + 2\|n\| - n(\hat{\tau}+\alpha)$$

i.e.

$$2\delta \ge R + 2\|n\| - n(\hat{\tau}+\alpha) + n(\frac{\tau_0+s_0}{2})$$

i.e.

$$\delta \ge R/2 \ .$$

Now we assert the following about (x_0,y_0,τ_0,s_0).

$$\left\{\begin{array}{l} \text{As} \quad \delta \downarrow 0 \ |x_0-y_0| \le \varepsilon, \ \tau_0,s_0 \to \hat{\tau} \ \text{and} \\ (u(x_0,\tau_0) - \bar{u}(y_0,s_0))^+ + 3Re^{-\gamma(\frac{\tau_0+s_0}{2})}\beta_\varepsilon(x_0-y_0) = (u(x_0,\tau_0) - \\ - \bar{u}(y_0,s_0)) + 3Re^{-\gamma(\frac{\tau_0+s_0}{2})}\beta_\varepsilon(x_0-y_0) \to m^+(\hat{\tau}) \ . \end{array}\right. \tag{1.17}$$

Indeed let δ be so small that

$$2\delta + |n(s) - n(t)| < R$$

for $|s-t| < \delta/2$. If $|x_0-y_0| > \varepsilon$, then (1.5), (1.15) and (1.16) imply

$$2R + 3R + 2\|n\| + 2\delta - n\left(\frac{\tau_0+s_0}{2}\right) \geq \Psi(x_0,y_0,\tau_0,s_0) \geq \Psi(x,x,\tau_0,\tau_0) \geq$$

$$\geq 3Re^{-\gamma\tau_0} + 3R + 2\|n\| - n(\tau_0)$$

i.e.
$$2\delta + n(\tau_0) - n\left(\frac{\tau_0+s_0}{2}\right) \geq R$$

which is a contradiction. Note that here is where we really used the assumption $\gamma < 0$. Moreover, suppose that as $\delta \to 0$ $\tau_0, s_0 \to \bar{\tau} \in I$ along a subsequence (which for simplicity is denoted in the same way as the sequence). Again (1.15), together with the facts that $u, \bar{u} \in BUC(\bar{Q}_T)$ and $|x_0-y_0| \leq \varepsilon$, implies that, for every $(x,y) \in R^N \times R^N$ and $\tau \in I$, it is

$$e^{-\frac{\gamma}{2}(\tau_0+s_0)}|\bar{u}(y_0,\tau_0) - \bar{u}(y_0,s_0)| + m^+(\tau_0) - n\left(\frac{\tau_0+s_0}{2}\right) + 2\delta +$$

$$+ 3R + 2\|n\| \geq \Psi(x_0,y_0,\tau_0,s_0) \geq \Psi(x,y,\tau,\tau) \geq 3R + 2\|n\| +$$

$$+ (u(x,\tau) - \bar{u}(y,\tau))^+ + 3Re^{-\gamma\tau}\beta_\varepsilon(x-y) - n(\tau)$$

i.e.
$$e^{-\frac{\gamma}{2}(\tau_0+s_0)}|\bar{u}(y_0,\tau_0) - \bar{u}(y_0,s_0)| + m^+(\tau_0) - n\left(\frac{\tau_0+s_0}{2}\right) + 2\delta \geq$$

$$\geq m^+(\tau) - n(\tau) \quad .$$

Letting $\delta \downarrow 0$ we get

$$m^+(\bar{\tau}) - n(\bar{\tau}) \geq m^+(\tau) - n(\tau) \quad \text{for every} \quad \tau \in I \quad .$$

But then $\bar{\tau} = \hat{\tau}$, since $\hat{\tau}$ is a strict maximum of $m^+ - n$ on I. Next observe that (1.15) and the fact that $\tau_0, s_0 \to \hat{\tau}$ as $\delta \to 0$ imply that

$$m^+(\hat{\tau}) - n(\hat{\tau}) \geq \overline{\lim_{\delta\downarrow 0}} \{(u(x_0,\tau_0) - \bar{u}(y_0,s_0))^+ + 3Re^{-\gamma\frac{\tau_0+s_0}{2}}\beta_\varepsilon(x_0-y_0)\} - n(\hat{\tau}) \geq$$

$$\geq \underline{\lim_{\delta\downarrow 0}} \{(u(x_0,\tau_0) - \bar{u}(y_0,s_0))^+ + 3Re^{-\gamma\frac{\tau_0+s_0}{2}}\beta_\varepsilon(x_0-y_0)\} - n(\hat{\tau}) \geq$$

$$\geq m^+(\hat{\tau}) - n(\hat{\tau})$$

i.e. $(u(x_0,\tau_0) - \bar{u}(y_0,s_0))^+ + 3Re^{-\gamma\frac{\tau_0+s_0}{2}}\beta_\varepsilon(x_0-y_0) + m^+(\hat{\tau})$.

Finally, if along some subsequence $\delta \downarrow 0$, it is

$$(u(x_0,\tau_0) - \bar{u}(y_0,s_0))^+ = 0$$

then

$$m(\hat{\tau}) \le 3Re^{-\gamma\hat{\tau}}$$

which contradicts (1.12).

Next observe that $(x_0,\tau_0) \in Q_T$ is a local maximum of $(x,\tau) \to u(x,\tau) + 3Re^{-\gamma\frac{\tau+s_0}{2}}\beta_\varepsilon(x-y_0) + (3R+2\|n\|)\gamma_\delta(\tau-s_0) + 2\delta\zeta(x,y_0) - n(\frac{\tau+s_0}{2})$ and $(y_0,s_0) \in Q_T$ is a local minimum of $(y,s) \to \bar{u}(y,s) - 3Re^{-\gamma\frac{\tau_0+s}{2}}\beta_\varepsilon(x_0-y) - (3R+2\|n\|)\gamma_\delta(\tau_0-s) - 2\delta\zeta(x_0,y) + n(\frac{\tau_0+s}{2})$. In view of (1.1) and (1.2) we have

$$(3R+2\|n\|)\gamma_\delta'(\tau_0-s_0) + \frac{1}{2}n'(\frac{\tau_0+s_0}{2}) + \frac{\gamma}{2}3Re^{-\gamma\frac{\tau_0+s_0}{2}}\beta_\varepsilon(x_0-y_0) +$$
$$+ H(\tau_0,x_0,u(x_0,\tau_0), - 3Re^{-\gamma\frac{\tau_0+s_0}{2}}D\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) \le 0$$

and

$$0 \le -(3R+2\|n\|)\gamma_\delta'(\tau_0-s_0) - \frac{1}{2}n'(\frac{\tau_0+s_0}{2}) - \frac{\gamma}{2}3Re^{-\gamma\frac{\tau_0+s_0}{2}}\beta_\varepsilon(x_0-y_0) +$$
$$+ \bar{H}(s_0,y_0,\bar{u}(y_0,s_0), - 3Re^{-\gamma\frac{\tau_0+s_0}{2}}D\beta_\varepsilon(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0)) .$$

Combining these two inequalities we obtain:

$$n'(\frac{\tau_0+s_0}{2}) + \gamma 3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}\beta_\varepsilon(x_0-y_0) \le \bar{H}(x_0,y_0,\bar{u}(y_0,s_0), -3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) +$$

$$2\delta D_y\zeta(x_0,y_0)) - H(\tau_0,x_0,u(x_0,\tau_0), -3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)).$$

To continue we assume that $\|\bar{u}\| = \min(\|u\|,\|\bar{u}\|)$. (If not then one has to modify the rest of the proof in an obvious way.) Then in view of (1.17) and (H3) and for δ small, we have

$$n'(\frac{\tau_0+s_0}{2}) + \gamma\{(u(x_0,\tau_0) - \bar{u}(y_0,s_0))^+ + 3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}\beta_\varepsilon(x_0-y_0)\} \leq$$

$$\leq \bar{H}(s_0,y_0,\bar{u}(y_0,s_0), -3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0)) -$$

$$- H(\tau_0,x_0,\bar{u}(y_0,s_0), -3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) \quad .$$

Next observe that for $\delta < 1/2$

$$|-3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0)|,$$

$$|-3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)| \leq \frac{6Re^{|\gamma|T}}{\varepsilon} + 1 \quad .$$

Moreover if $L < \infty$ and (without any loss of generality) $L = \sup_{0\leq t\leq T}\|Du(\cdot,\tau)\|$, then, since x_0 is a maximum point of the mapping $x \to u(x,\tau_0) + 3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}\beta_\varepsilon(x-y_0) + 2\delta\zeta(x,y_0)$, for $x \in \mathbb{R}^N$ we have

$$3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}\beta_\varepsilon(x-y_0) + 2\delta\zeta(x,y_0) - 3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}\beta_\varepsilon(x_0-y_0) - 2\delta\zeta(x_0,y_0) \leq$$

$$\leq L|x-x_0| \quad .$$

But this implies that

$$|3Re^{-\frac{\gamma}{2}(\tau_0+s_0)} D\beta_\varepsilon(x_0-y_0) + 2\delta D_x\zeta(x_0,y_0)| \le L \quad .$$

Combining all the above we obtain

$$n'(\frac{\tau_0+s_0}{2}) + \gamma\{(u(x_0,\tau_0) - \bar{u}(y_0,s_0))^+ + 3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}\beta_\varepsilon(x_0-y_0)\} \le$$

$$\le \bar{H}(\tau_0,y_0,\bar{u}(y_0,s_0), -3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0))$$

$$- H(\tau_0,x_0,\bar{u}(y_0,s_0), -3Re^{-\frac{\gamma}{2}(\tau_0+s_0)}D\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0))$$

$$+ \omega_{\bar{H},\, \max(\frac{6Re^{|\gamma|T}}{\varepsilon} + 1, R)}(4\delta)$$

$$\le \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p) - \bar{H}(t,y,r,p)| + \omega_{\bar{H},\, \max(\frac{6Re^{|\gamma|T}}{\varepsilon} + 1, R)}(4\delta) \quad .$$

where for $\bar{R} > 0$, $\omega_{\bar{H},\bar{R}}(\alpha)$ denotes the modulus of continuity of $\bar{H}$ on $[0,T] \times \mathbf{R}^N \times [-\bar{R},\bar{R}] \times B_N(0,\bar{R})$. Letting $\delta \downarrow 0$ in the last inequality we get (1.11).

Next we use Proposition 1.4 to establish several properties of the viscosity solution $u \in BUC(\bar{Q}_T)$ of (0.2). In particular, we describe the evolution in time of the norm, the modulus of continuity (in the x variable) and the Lipschitz constant (in the x variable) if $u(\cdot,\tau) \in C_b^{0,1}(\mathbf{R}^N)$ for $\tau \in [0,T]$. Moreover we give an estimate for $\|u(\cdot,\tau) - u_0\|$ in the case that $u_0 \in C_b^{0,1}(\mathbf{R}^N)$. Before we state the results we introduce a notation for the modulus of continuity of a function $f : \mathcal{O} \to \mathbf{R}$. It is

(1.18) $$\omega_f(r) = \sup_{|x-y|\le r} |f(x) - f(y)| \quad .$$

<u>Proposition 1.5</u>: Let $H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1) and (H3) with $\gamma_R \leq 0$ for every $R > 0$. If for $u_0 \in BUC(\mathbb{R}^N)$, $u \in BUC(\bar{Q}_T)$ is a viscosity solution of (0.2), let $R > \|u\|$ and $\gamma = \gamma_R$. The following are true

(a) If H satisfies (H2), then

$$\|u(\cdot,\tau)\| \leq e^{-\gamma\tau}(\tau C + \|u_0\|) \quad \text{for every} \quad \tau \in [0,T] \tag{1.19}$$

where C is given by (H2).

(b) If H satisfies (H4), then for $1 > r > 0$

$$\omega_{u(\cdot,\tau)}(r) \leq e^{-\gamma\tau}(2\omega_{u_0}(r) + \tau\Lambda_{12Re^{|\gamma|T}+2}(2r)) \quad \text{for every} \quad \tau \in [0,T] \tag{1.20}$$

(c) If H satisfies (H5) and for every $\tau \in [0,T]$, $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ with $L = \sup_{0\leq\tau\leq T} \|Du(\cdot,\tau)\|$, then for every $\tau \in [0,T]$

$$\|Du(\cdot,\tau)\| \leq e^{-\gamma\tau}(L_0 + \tau[C_R(1+L)]) \tag{1.21}$$

where $L_0 = \|Du_0\|$ and C_R is given by (H5). Moreover

$$L \leq e^{T(2C_R e^{-\gamma T} - \gamma)}(L_0 + TC_R) \tag{1.22}$$

(d) If $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then

$$\|u(\cdot,\tau) - u_0\| \leq \tau e^{-\gamma\tau} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\leq\|u_0\| \\ |p|\leq\|Du_0\|}} |H(t,x,r,p)| \quad \text{for every} \quad \tau \in [0,T] \tag{1.23}$$

(e) If for every $\tau \in [0,T]$, $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ and $\sup_{0\leq\tau\leq T} \|Du(\cdot,\tau)\| \leq L$, then $u \in C_b^{0,1}(\bar{Q}_T)$ and for $t,s \in [0,T]$

$$(1.24) \qquad \|u(\cdot,\tau) - u(\cdot,s)\| \le |\tau-s| e^{-\gamma T} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le\|u\| \\ |p|\le L}} |H(t,x,r,p)| \quad .$$

Proof. (a) We apply Proposition 1.4 to u and $\bar{u} = 0$ which is an obvious viscosity solution of the problem

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} + 0 = 0 & \text{in } Q_T \\ \bar{u}(x,0) = 0 & \text{in } \mathbf{R}^N \quad . \end{cases}$$

Then, for $\tau \in [0,T]$ and $\varepsilon > 0$, (1.7) implies

$$\|u(\cdot,\tau)\| + 3Re^{-\gamma\tau} \le \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau)| + 3Re^{-\gamma\tau}\beta_\varepsilon(x-y)\} \le$$

$$\le e^{-\gamma\tau} \sup_{(x,y)\in D_\varepsilon} \{|u_0(x)| + 3Re^{-\gamma\tau}\} + e^{-\gamma\tau}\tau \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p)| \quad .$$

But in this case

$$A_\varepsilon = \{(t,x,y,r,p) : t \in [0,T], |x-y| \le \varepsilon, |r| \le \min(\|u\|,0),$$

$$|p| \le \min(\frac{6Re^{|\gamma|T}}{\varepsilon} + 1,0)\} = \{(t,x,y,0,0) : t \in [0,T], |x-y| \le \varepsilon\} \quad .$$

So

$$\sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p)| = \sup_{(t,x)\in\bar{Q}_T} |H(t,x,0,0)| = C \quad .$$

This implies (1.19).

(b) For $1 > r > 0$ fixed, let $\xi \in \mathbf{R}^N$ be such that

$$|\xi| \le r \quad .$$

If $\bar{u} : \bar{Q}_T \to \mathbf{R}$ is defined by

$$\bar{u}(x,\tau) = u(x+\xi,\tau)$$

then obviously $\bar{u} \in BUC(\bar{Q}_T)$. Moreover $\bar{u}$ is a viscosity solution of

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} + H(t,x+\xi,\bar{u},D\bar{u}) = 0 & \text{in } Q_T \\ \bar{u}(x,0) = u_0(x+\xi) & \text{in } \mathbb{R}^N \end{cases} .$$

To see this, we have to check (1.1) and (1.2). Here we only prove (1.1), since the proof of (1.2) is identical. To this end, observe that, if for $\phi \in C^\infty(Q_T)$, $(x_0,\tau_0) \in Q_T$ is a local maximum of $\bar{u} - \phi$, then $(x_0+\xi,\tau_0)$ is a local maximum of $u - \psi$, where $\psi(y,\tau) = \phi(y-\xi,\tau)$. By (1.1) we have

$$\frac{\partial \psi}{\partial t}(x_0+\xi,\tau_0) + H(\tau_0,x_0+\xi,u(x_0+\xi,\tau_0),D\psi(x_0+\xi,\tau_0)) \le 0$$

i.e.

$$\frac{\partial \phi}{\partial t}(x_0,\tau_0) + H(\tau_0,x_0+\xi,\bar{u}(x_0,\tau_0),D\phi(x_0,\tau_0)) \le 0 \quad .$$

Now applying Proposition 1.4 to u, $\bar{u}$ for $\tau \in [0,T]$ and $\varepsilon = r$ we have

$$\sup_{x\in R^N}|u(x,\tau)-u(x+\xi,\tau)| + 3Re^{-\gamma\tau} \le \sup_{(x,y)\in D_r}\{|u(x,\tau)-u(y+\xi,\tau)| + 3Re^{-\gamma\tau}\beta_r(x-y)\} \le$$

$$\le e^{-\gamma\tau}\sup_{(x,y)\in D_r}|u_0(x)-u_0(y+\xi)| + 3Re^{-\gamma\tau} + e^{-\gamma\tau}\tau \sup_{(t,x,y,s,p)\in A_r}|H(t,x,s,p) - H(t,y+\xi,s,p)| \quad .$$

But in view of (1.18) and (H4)

$$\sup_{(x,y)\in D_r}|u_0(x) - u_0(y+\xi)| \le \omega_{u_0}(r+|\xi|) \le 2\omega_{u_0}(r)$$

and

$$\sup_{(t,x,y,s,p)\in A_r}|H(t,x,s,p) - H(t,y+\xi,s,p)| \le \sup_{\substack{t\in[0,T]\\|x-y|\le r\\|s|\le R\\|p|\le\frac{6Re^{|\gamma|}}{r}+1}}\{|H(t,x,s,p) - H(t,y+\xi,s,p)|\} \le \Lambda_{12Re^{|\gamma|T}+1}(2r)$$

thus the result.

(c) For $\xi \in \mathbb{R}^N$ define $\bar{u} : \bar{Q}_T \to \mathbb{R}$ by

$$\bar{u}(x,\tau) = u(x+\xi,\tau) \quad .$$

Then $\bar{u} \in BUC(\bar{Q}_T)$, $\bar{u}(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$ for every $\tau \in BUC(\mathbb{R}^N)$ and as shown in (b), $\bar{u}$ is a viscosity solution of

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + H(t,x+\xi,\bar{u},D\bar{u}) = 0 & \text{in } Q_T \\ \bar{u}(x,0) = u_0(x) & \text{in } \mathbb{R}^N \quad . \end{cases}$$

Applying Proposition 1.4 to u, $\bar{u}$ for $\tau \in [0,T]$ and $\varepsilon > 0$ we have

$$\sup_x |u(x,\tau)-u(x+\xi,\tau)| + 3Re^{-\gamma\tau} \le \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau)-u(y+\xi,\tau)| + 3Re^{-\gamma\tau}\beta_\varepsilon(x-y)\} \le$$

$$\le e^{-\gamma\tau}\Big(\sup_{(x,y)\in D_\varepsilon} |u_0(x) - u_0(y+\xi)|\Big) + 3Re^{-\gamma\tau} +$$

$$+ e^{-\gamma\tau}\tau \sup_{(t,x,y,s,p)\in A_\varepsilon} |H(t,x,r,p) - H(t,y+\xi,r,p)|$$

and therefore

$$\sup_x |u(x,\tau)-u(x+\xi,\tau)| \le e^{-\gamma\tau}L_0(\varepsilon+|\xi|) +$$

$$+ e^{-\gamma\tau}\tau \sup_{(t,x,y,s,p)\in A_\varepsilon} |H(t,x,s,p) - H(t,y+\xi,s,p)| \quad .$$

But in view of the definition of A_ε and (H5) we have

$$\sup_{(t,x,y,s,p)\in A_\varepsilon} |H(t,x,s,p) - H(t,y+\xi,s,p)| \le C_R(1+L)(\varepsilon+|\xi|) \quad .$$

Combining the above and letting $\varepsilon \to 0$ we get

$$\sup_x |u(x,\tau) - u(x+\xi,\tau)| \le e^{-\gamma\tau}[L_0 + \tau C_R(1+L)]|\xi|$$

and thus (1.21).

To prove the second part of the claim, we choose a positive integer m so that

$$0 < \frac{TC_R}{m} e^{-\gamma T} < \frac{1}{2} .$$

For $i = 1,\dots,m$ let Q_i, $\bar{Q}_i$, u_i and L_i be defined by

$$Q_i = \mathbb{R}^N \times (\frac{i-1}{m} T, \frac{iT}{m}]$$

$$\bar{Q}_i = \mathbb{R}^N \times [\frac{i-1}{m} T, \frac{iT}{m}]$$

$$u_i = u|_{\bar{Q}_i}$$

and

$$L_i = \sup_{\tau \in [\frac{i-1}{m} T, \frac{i+1}{m} T]} \|Du(\cdot,\tau)\|$$

where for $f : \mathcal{O} \to \mathbb{R}$ and C a subset of $\mathcal{O}$, $f|_C$ denotes the restriction of f on C. Then $u_i \in BUC(\bar{Q}_i)$ is a viscosity solution of

$$\begin{cases} \frac{\partial u_i}{\partial t} + H(t,x,u_i,Du_i) = 0 \quad \text{in } Q_i \\ u_i(x, \frac{i-1}{m} T) = u(x, \frac{i-1}{m} T) \quad \text{in } \mathbb{R}^N \end{cases}$$

where the obvious extension of Definition 1.1 has been assumed here. Applying the first part of the claim to u_i we obtain

$$L_i \le e^{-\gamma \frac{T}{m}}(L_{i-1} + C_R \frac{T}{m}(1+L_i))$$

i.e. $$L_i \le e^{-\gamma \frac{T}{m}}(L_i + C_R \frac{T}{m}(1+L_i))$$

i.e. $$L_i \le \frac{e^{-\gamma T/m}}{1-C_R \frac{T}{m} e^{-\gamma T}} (L_{i-1} + C_R \frac{T}{m})$$

i.e. $$L_i \le e^{(2C_R \frac{T}{m} e^{-\gamma T} - \gamma \frac{T}{m})} (L_{i-1} + C_R\, T/m)$$

where here we used the fact that for $0 \le x \le \frac{1}{2}$

$$\frac{1}{1-x} \le e^{x+x^2} \le e^{2x} .$$

A simple inductive argument implies (1.22).

(d) Applying Proposition 1.4 to $u \in BUC(\bar{Q}_T)$ and $u_0 \in BUC(\mathbb{R}^N)$, which is an obvious viscosity solution of

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} + 0 = 0 & \text{in } Q_T \\ \bar{u}(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

for $\tau \in [0,T]$ and $\varepsilon > 0$ we have

$$\sup_x |u(x,\tau)-u_0(x)| + 3Re^{-\gamma\tau} \le \sup_{(x,y)\in D_\varepsilon} \{|u(x,\tau)-u_0(y)| + $$

$$+ 3Re^{-\gamma\tau}\beta_\varepsilon(x-y)\}$$

$$\le e^{-\gamma\tau} \sup_{(x,y)\in D_\varepsilon} |u_0(x)-u_0(y)| + 3Re^{-\gamma\tau} + e^{-\gamma\tau} \sup_{(\tau,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p)|$$

therefore

$$\|u(\cdot,\tau)-u_0\| \le e^{-\gamma\tau} L_0 \varepsilon + e^{-\gamma\tau} \sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p)| .$$

But in view of the definition of A_ε we have

$$\sup_{(t,x,y,r,p)\in A_\varepsilon} |H(t,x,r,p)| \le \sup_{\substack{(x,t)\in \bar{Q}_T \\ |r|\le \|u_0\| \\ |p|\le L_0}} |H(t,x,r,p)|$$

and thus the result

(e) For any $s \in [0,T]$, u is the viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial \tau} + H(\tau,x,u,Du) = 0 & \text{in } \mathbb{R}^N \times (s,T] \\ u(x,s) = u(x,s) & \text{in } \mathbb{R}^N \end{cases}$$

as one can easily check. Then (d), for $\tau \in [s,T]$, implies

$$\|u(\cdot,\tau)-u(\cdot,s)\| \leq (\tau-s)e^{-\gamma\tau} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\leq\|u\| \\ |p|\leq L}} |H(t,x,r,p)|$$

and thus the result.

Section 2

We begin this section with a result concerning the existence of the viscosity solution of (0.2), in the case that H and u_0 are sufficiently smooth functions. In particular, we show that the solution of the viscosity approximation

$$(2.1)_\varepsilon \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \varepsilon \Delta u_\varepsilon + H(t,x,u_\varepsilon,Du_\varepsilon) = 0 & \text{in } Q_T \\ u_\varepsilon(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

converges as $\varepsilon \to 0$ uniformly in $\bar{Q}_T$ to a function u which is then, by Proposition 1.2, the viscosity solution of (0.2). Moreover, we give an explicit estimate on $\|u-u_\varepsilon\|$.

Proposition 2.1. Let $H \in C_b^2([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfy (H2), (H3) with $\gamma = \gamma_R < 0$ for every $R > 0$ and (H5). For $u_0 \in C_b^2(\mathbb{R}^N)$ and $\varepsilon > 0$, let $u_\varepsilon \in BUC(\mathbb{R}^N) \cap C^{2,1}(Q_T)^{(*)}$ be the solution of $(2.1)_\varepsilon$. Then there exists $u \in BUC(\mathbb{R}^N)$ such that u is the viscosity solution of (0.2) and $u_\varepsilon \to u$ uniformly on $\bar{Q}_T$ as $\varepsilon \to 0$. Moreover, it is

$$\sup_{0<\tau<T} \|u_\varepsilon(\cdot,\tau) - u(\cdot,\tau)\| < K\sqrt{\varepsilon} \tag{2.2}$$

where K is a constant which depends only on $\|u_0\|$ and $\|Du_0\|$.

Remark 2.1. M. G. Crandall and P. L. Lions proved the above result in [3] for the case of (0.4). Moreover estimates like (2.2) have also been obtained by W. H. Fleming ([5]) and P. L. Lions ([7]) by indirect arguments involving stochastic differential games.

(*) $u \in C^{2,1}(Q_T)$ means that $\dfrac{\partial u}{\partial x_i \partial x_j}, \dfrac{\partial u}{\partial t} \in C(Q_T)$

Proof of Proposition 2.1. The existence of such an u_ε follows from standard theory. (See in particular A. Friedman [6].) Moreover, it is also known that under our assumptions on H, u_0 and for every $\tau \in [0,T]$, $u_\varepsilon(\cdot,\tau) \in C_b^{0,1}(\mathbb{R}^N)$. In order to prove the existence of u it suffices to show that as $\varepsilon \to 0$ $\{u_\varepsilon\}$ forms a Cauchy family in $BUC(\bar{Q}_T)$. Indeed then there exists $u \in BUC(\bar{Q}_T)$ such that $u_\varepsilon \to u$ uniformly in $\bar{Q}_T$ as $\varepsilon \to 0$. By Proposition 1.2 and theorem 1.1 u is the viscosity solution of (0.2). To this end, we show that there exists a constant K, which depends only on $\|Du_0\|$ and $\|u_0\|$, such that for $\varepsilon, \eta > 0$

$$\sup_{0\le\tau\le T} \|u_\varepsilon(\cdot,\tau)-u_\eta(\cdot,\tau)\| \le K(\sqrt{\varepsilon} + \sqrt{\eta}) \quad . \tag{2.3}$$

To prove (2.3) we need the following lemma:

Lemma 2.1. If H, u_0, ε and u_ε are as in Proposition 2.1, then for every $\tau \in [0,T]$

$$\|u_\varepsilon(\cdot,\tau)\| \le e^{-\gamma\tau}(\|u_0\| + C\tau) \tag{2.4}$$

where C is given by (H2) and

$$\|Du_\varepsilon(\cdot,\tau)\| \le e^{-\gamma\tau}(\|Du_0\| + \tau C_R(1 + L_\varepsilon)) \tag{2.5}$$

where $L_\varepsilon = \sup_{0\le\tau\le T} \|Du_\varepsilon(\cdot,\tau)\|$ and $R > e^{-\gamma T}(\|u_0\| + CT)$. Moreover L_ε satisfies

$$L_\varepsilon \le e^{T(2C_R e^{-\gamma T}-\gamma)}(\|Du_0\| + TC_R) = \bar{L} \quad . \tag{2.6}$$

We first complete the proof of the proposition and then prove the lemma. Observe that it suffices to show that there is a constant K, which depends only on $\|u_0\|$, $\|Du_0\|$, such that for $\varepsilon, \eta > 0$

$$\sup_{0\le\tau\le T}\ \sup_{x\in\mathbb{R}^N} (u_\varepsilon(x,\tau) - u_\eta(x,\tau))^+ \le K(\sqrt{\varepsilon} + \sqrt{\eta}) \quad . \tag{2.7$^\pm$}$$

Here we establish only $(2.4)^+$ since $(2.4)^-$ can be proved in exactly the same way. To this end and for $\theta = \sqrt[4]{\varepsilon} + \sqrt[4]{\eta}$ and $R > e^{-\gamma T}(\|u_0\| + CT)$, let

$m : [0,T] \to \mathbf{R}$ be defined by

$$(2.8) \qquad m(\tau) = \sup_{|x-y|\le \bar{L}\theta^2} \{(u_\varepsilon(x,\tau) - u_\eta(y,\tau))^+ + 3(R+1)e^{-\gamma\tau}\beta_\theta(x-y)\}$$

where $\beta_\theta(w) = \beta(\frac{w}{\theta})$, with $\beta \in C_0^\infty(R^N)$ such that

$$(2.9) \qquad \begin{cases} \beta(0) = 1,\ 0 \le \beta \le 1,\ \beta(w) = 0 \ \text{ if } \ |w| > 1 \\ \beta(w) = 1 - |w|^2 \ \text{ for } \ |w| \le \frac{\sqrt{3}}{2} \\ \text{and} \\ \beta(w) < 1/2 \ \text{ for } \ |w| > \frac{\sqrt{3}}{2} \end{cases}$$

and R, $\bar{L}$ are given by (2.6). We claim that m, which is a continuous function, is a solution of the viscosity inequality

$$(2.10) \qquad m'(\tau) + \gamma m(\tau) \le K_1(\sqrt{\varepsilon} + \sqrt{\eta})$$

where K_1 depends only on $\|u_0\|$ and $\|Du_0\|$. Before we prove this claim, we show that it implies $(2.7)^+$. Indeed in view of Proposition 1.1, Remark 1.3 and the fact that $\gamma \le 0$, for every $\tau \in [0,T]$, it is

$$m(\tau) \le e^{-\gamma\tau}(m(0) + \tau K_1(\sqrt{\varepsilon} + \sqrt{\eta})) \quad .$$

But then

$$\sup_x (u_\varepsilon(x,\tau)-u_\eta(x,\tau))^+ + 3(R+1)e^{-\gamma\tau} \le \sup_{|x-y|\le\bar{L}\theta^2} \{(u_\varepsilon(x,\tau)-u_\eta(y,\tau))^+ +$$

$$+ 3(R+1)e^{-\gamma\tau}\beta_\theta(x-y)\} \le e^{-\gamma\tau} \sup_{|x-y|\le\bar{L}\theta^2} |u_0(x)-u_0(y)| + 3(R+1)e^{-\gamma\tau} +$$

$$+ e^{-\gamma\tau}\tau K_1(\sqrt{\varepsilon} + \sqrt{\eta}) \le e^{-\gamma\tau}(2(\bar{L})^2 + \tau K_1)(\sqrt{\varepsilon} + \sqrt{\eta}) + 3Re^{-\gamma\tau}$$

since $\theta^2 = (\sqrt[4]{\varepsilon} + \sqrt[4]{\eta})^2 \le 2(\sqrt{\varepsilon} + \sqrt{\eta})$, and therefore

$$(2.11) \qquad \sup_x (u_\varepsilon(x,\tau) - u_\eta(x,\tau))^+ \le e^{-\gamma\tau}(2(\bar{L})^2 + \tau K_1)(\sqrt{\varepsilon} + \sqrt{\eta}) \quad .$$

For the proof of the claim, let $n \in C^\infty((0,T))$ and assume that $\hat{\tau} \in (0,T)$ is a strict local maximum of $m - n$ on $I = [\hat{\tau}-\alpha, \hat{\tau}+\alpha] \subset (0,T)$ for some

$\alpha > 0$. We are going to show that

$$n'(\hat{\tau}) + \gamma m(\hat{\tau}) \leq K_1(\sqrt{\varepsilon} + \sqrt{\eta}) \tag{2.12}$$

and thus, in view of *Remark 1.3, (2.10). If* $m(\hat{\tau}) = 3(R+1)e^{-\gamma\hat{\tau}}$, then, for every $\tau \in I$, we have

$$3(R+1)e^{-\gamma\hat{\tau}} - \eta(\hat{\tau}) \geq m(\tau) - n(\tau) \geq 3(R+1)e^{-\gamma\tau} - \eta(\tau)$$

i.e.

$$n'(\hat{\tau}) = -\gamma 3(R+1)e^{-\gamma\hat{\tau}}$$

i.e.

$$n'(\hat{\tau}) + \gamma m(\hat{\tau}) = 0$$

and thus (2.12). Now we assume that

$$m(\hat{\tau}) > 3(R+1)e^{-\gamma\hat{\tau}} \tag{2.13}$$

and we define $\Phi : R^N \times R^N \times I \to R$ by

$$\Phi(x,y,\tau) = (u_\varepsilon(x,\tau) - u_\eta(y,\tau))^+ + 3(R+1)e^{-\gamma\tau}\beta_\theta(x-y) - n(\tau) \quad . \tag{2.14}$$

Since Φ is bounded on $R \times R^N \times I$, for every $\delta > 0$ there is a point $(x_1,y_1,\tau_1) \in R^N \times R^N \times I$ such that

$$\Phi(x_1,y_1,\tau_1) > \sup_{R^N\times R^N\times I} \Phi - \delta \quad .$$

Next select $\zeta \in C_0^\infty(R^N\times R^N)$ satisfying $0 \leq \zeta \leq 1$, $\zeta(x_1,y_1) = 1$, $|D\zeta| \leq 1$, $|\Delta\zeta| \leq 1$ and define $\Psi : R^N \times R^N \times I \to R$ by

$$\Psi(x,y,\tau) = \Phi(x,y,\tau) + 2\delta\zeta(x,y) \quad . \tag{2.15}$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1,\tau_1) = \Phi(x_1,y_1,\tau_1) + 2\delta$$

there exists a point $(x_0,y_0,\tau_0) \in R^N \times R^N \times I$ such that

$$\Psi(x_0,y_0,\tau_0) \geq \Psi(x,y,\tau) \quad \text{for every } (x,y,\tau) \in R^N \times R^N \times I \quad . \tag{2.16}$$

We assert the following about (x_0,y_0,τ_0)

$$\left\{\begin{array}{l} \text{For } \delta < \min(\frac{1}{8}, \bar{L}^{-2}\theta^2),\ |x_0-y_0| \leq (\bar{L} + 2\delta)\theta^2 \text{ and as } \delta \downarrow 0 \\ \tau_0 \to \hat{\tau} \text{ and } (u_\varepsilon(x_0,\tau_0) - u_\eta(y_0,\tau_0))^+ + 3Re^{-\gamma\tau_0}\beta_\theta(x_0-y_0) = \\ = u_\varepsilon(x_0,\tau_0) - u_\eta(y_0,\tau_0) + 3Re^{-\gamma\tau_0}\beta_\theta(x_0-y_0) + m(\hat{\tau}) \quad . \end{array}\right. \tag{2.17}$$

Indeed (2.9) and (2.16) together with the fact that $\gamma < 0$ imply that

$$2(R+1)e^{-\gamma\tau_0} + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0) + 2\delta - n(\tau_0) \geq (u_\varepsilon(x_0,\tau_0) - u_\eta(y_0,\tau_0))^+ +$$

$$+ 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0) + 2\delta\zeta(x_0,y_0) - n(\tau_0) =$$

$$= \Psi(x_0,y_0,\tau_0) \geq \Psi(x,x,\tau_0) \geq 3(R+1)e^{-\gamma\tau_0} - n(\tau_0)$$

i.e.
$$\beta_\theta(x_0-y_0) \geq \frac{1}{3} - \frac{2\delta}{3(R+1)e^{-\gamma\tau_0}} \geq \frac{1}{3} - \frac{2\delta}{3} > \frac{1}{4} .$$

Thus in view of (2.9) $|x_0-y_0| \leq \theta$ and

(2.18)
$$\beta_\theta(x_0-y_0) = 1 - \frac{|x_0-y_0|^2}{\theta^2} .$$

Moreover if $(u_\varepsilon(x_0,\tau_0) - u_\eta(y_0,\tau_0))^+ = 0$, the above inequalities give

$$\beta_\theta(x_0-y_0) \geq 1 - \frac{2\delta}{3} \geq 1 - \bar{L}^{-2}\theta^2$$

and therefore

$$|x_0-y_0| \leq \bar{L}\theta^2 .$$

So we may assume that $(u_\varepsilon(x_0,\tau_0) - u_\eta(y_0,\tau_0))^+ > 0$. In this case, in view of the fact that because of (2.16) x_0 is a maximum point of $x \mapsto u_\varepsilon(x,\tau_0) + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x-y_0) + 2\delta\zeta(x,y_0)$, for every $x \in \mathbb{R}^N$, we have

$$3(R+1)e^{-\gamma\tau_0}\beta_\theta(x-y_0) + 2\delta\zeta(x,y_0) - 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0) - 2\delta\zeta(x_0,y_0) \leq$$

$$\leq u(x_0,\tau_0) - u(x,\tau_0) \leq \bar{L}|x-x_0| .$$

Therefore

$$|3(R+1)e^{-\gamma\tau_0}D\beta_\theta(x_0-y_0) + 2\delta D_x\zeta(x_0,y_0)| \leq \bar{L}$$

i.e.
$$3(R+1)e^{-\gamma\tau_0}|D\beta_\theta(x_0-y_0)| \leq \bar{L} + 2\delta \leq 6(R+1)e^{-\gamma\tau_0}(\bar{L}+2\delta)$$

and by (2.18)

$$6(R+1)e^{-\gamma\tau_0}\frac{|x_0-y_0|}{\theta^2} \leq 6(R+1)e^{-\gamma\tau_0}(\bar{L}+2\delta)$$

i.e. $$|x_0-y_0| \leq (\bar{L}+2\delta)\theta^2 \ .$$

Now suppose that as $\delta \downarrow 0$ $\tau_0 \to \bar{\tau} \in I$ along a subsequence (which for simplicity is denoted in the same way as the sequence). For each δ for which $|x_0-y_0| > \bar{L}\theta^2$, we choose $\bar{y}_0 \in R^N$ such that

$$(2.19)\qquad \begin{cases} |x_0-y_0| = |x_0-\bar{y}_0| + |\bar{y}_0-y_0| \\ \text{and} \\ |x_0-y_0| = \bar{L}\theta^2 \ . \end{cases}$$

If $|x_0-y_0| \leq \bar{L}\theta^2$ let $y_0 = \bar{y}_0$. In either case and for $\delta < \min(\frac{1}{8}, \bar{L}^2\theta^2)$ it is

$$|x_0-\bar{y}_0| \leq \bar{L}\theta^2 \quad \text{and} \quad |y_0-\bar{y}_0| \leq 2\delta\theta^2 \ .$$

So, in view of (2.16) and the above observation, we have that for every $(x,y,\tau) \in R^N \times R^N \times I$,

$$\begin{aligned}&(u_\varepsilon(x_0,\tau_0)-u_\eta(\bar{y}_0,\tau_0))^+ + 3(R+1)e^{-\gamma\tau_0}\beta_0(x_0-\bar{y}_0) + \\ &\quad + \omega_{u_\eta}(|y_0-\bar{y}_0|) + 3(R+1)\omega_{\beta_\theta}(|y_0-\bar{y}_0|) + 2\delta - n(\tau_0) \geq \\ &\quad \geq \Psi(x_0,y_0,\tau_0) \geq \Psi(x,y,\tau) \geq (u_\varepsilon(x,\tau) - u_\eta(y,\tau))^+ + 3(R+1)e^{-\gamma\tau}\beta_\theta(x-y) - n(\tau)\end{aligned}$$

and therefore

$$m(\tau_0) - n(\tau_0) + \omega_{u_\eta}(2\delta\theta^2) + 3(R+1)e^{-\gamma\tau_0}\omega_{\beta_\theta}(2\delta\theta^2) + 2\delta \geq$$

$$(2.20)\qquad \geq (u_\varepsilon(x_0,\tau_0)-u_\eta(y_0,\tau_0))^+ + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0) +$$

$$+ 2\delta\zeta(x_0y_0) - n(\tau_0) \geq m(\tau) - n(\tau) \ .$$

Letting $\delta \downarrow 0$ in the above inequality we obtain

$$m(\bar{\tau}) - n(\bar{\tau}) \geq m(\tau) - n(\tau) \quad \text{for every } \tau \in I$$

which, in view of the definition of $\bar{\tau}$, implies

$$\bar{\tau} = \hat{\tau} \ .$$

This proves that as $\delta \downarrow 0$ $\tau_0 \to \hat{\tau}$. Moreover in this case, (2.20) also implies

$$m(\hat{\tau}) - n(\hat{\tau}) \geq \overline{\lim_{\delta\downarrow 0}} \{(u_\varepsilon(x_0,\tau_0)-u_\eta(y_0,\tau_0))^+ + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0)\} - n(\hat{\tau}) \geq$$

$$\geq \underline{\lim_{\delta\downarrow 0}} \{(u_\varepsilon(x_0,\tau_0)-u_\eta(y_0,\tau_0))^+ + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0)\} - n(\hat{\tau}) \geq$$

$$\geq m(\hat{\tau}) - n(\hat{\tau})$$

i.e. $$\lim_{\delta\downarrow 0} \{(u_\varepsilon(x_0,\tau_0)-u_\eta(y_0,\tau_0))^+ + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0)\} = m(\hat{\tau}) \quad .$$

Finally, for the last claim of assertion (2.17) observe that if along some subsequence $\delta \downarrow 0$

$$(u_\varepsilon(x_0,\tau_0) - u_\eta(y_0,\tau_0))^+ = 0$$

then

$$m(\hat{\tau}) \leq 3(R+1)e^{-\gamma\hat{\tau}}$$

which contradicts (2.13).

Next observe that, for δ sufficiently small, τ_0 is an interior maximum point of $\tau \to u_\varepsilon(x_0,\tau) - u_\eta(y_0,\tau) + 3(R+1)e^{-\gamma\tau}\beta_\theta(x_0-y_0) - n(\tau)$ in I, therefore

$$\frac{\partial u_\varepsilon}{\partial t}(x_0,\tau_0) - \frac{\partial u_\eta}{\partial t}(y_0,\tau_0) - \gamma 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0) - n'(\tau_0) = 0 \quad .$$

Moreover, x_0 is a maximum point of $x \to u_\varepsilon(x,\tau_0) + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x-y_0) + 2\delta\zeta(x,y_0)$ in $\mathbb{R}^N$ and y_0 is a minimum point of $y \to u_\eta(y,\tau_0) - 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y) - 2\delta\zeta(x_0,y)$ in $\mathbb{R}^N$. Therefore

$$Du_\varepsilon(x_0,\tau_0) = -3(R+1)e^{-\gamma\tau_0}D\beta_\theta(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)$$

$$Du_\eta(y_0,\tau_0) = -3(R+1)e^{-\gamma\tau_0}D\beta_\theta(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0)$$

and

$$\Delta u_\varepsilon(x_0,\tau_0) + 3(R+1)e^{-\gamma\tau_0}\Delta\beta_\theta(x_0-y_0) + 2\delta\Delta_x\zeta(x_0,y_0) \le 0$$

$$\Delta u_\eta(x_0,\tau_0) - 3(R+1)e^{-\gamma\tau_0}\Delta\beta_\theta(x_0-y_0) - 2\delta\Delta_y\zeta(x_0,y_0) \ge 0$$

where $\Delta_x\zeta(x_0,y_0) = \sum_{i=1}^{N} \frac{\partial^2\zeta}{\partial x_i^2}(x_0,y_0)$ and $\Delta_y\zeta(x_0,y_0) = \sum_{i=1}^{N} \frac{\partial^2\zeta}{\partial y_i^2}(x_0,y_0)$. The above, together with the fact that u_ε, u_η are solutions of $(2.1)_\varepsilon$, $(2.1)_\eta$ respectively, imply

$$n'(t_0) + \gamma 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0) \le -3(R+1)e^{-\gamma\tau_0}\Delta\beta_\theta(x_0-y_0)(\varepsilon+\eta) + 2\delta(\varepsilon+\eta) +$$

$$+ H(\tau_0,y_0,u_\eta(y_0,\tau_0), - 3(R+1)e^{-\gamma\tau_0}D\beta_\theta(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0)) -$$

$$- H(\tau_0,x_0,u_\varepsilon(x_\eta,\tau_0), - 3(R+1)e^{-\gamma\tau_0}D\beta_\theta(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) \quad .$$

In view of (H3), (H5), (2.9), (2.17) and (2.18), we have

$$n'(\tau_0) + \gamma\{(u_\varepsilon(x_0,\tau_0) - u_\eta(y_0,\tau_0))^+ + 3(R+1)e^{-\gamma\tau_0}\beta_\theta(x_0-y_0)\}$$

$$\le H(\tau_0,y_0,u_\varepsilon(x_0,\tau_0), - 3(R+1)e^{-\gamma\tau_0}D\beta_\theta(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0))$$

$$- H(\tau_0,x_0,u_\varepsilon(x_0,\tau_0), - 3(R+1)e^{-\gamma\tau_0}D\beta_\theta(x_0-y_0) + 2\delta D_x\zeta(x_0,y_0))$$

$$+ \omega_{H,3(R+1)e^{-\gamma T}\|D\beta_\theta\|+1}(4\delta) + 3(R+1)e^{-\gamma\tau_0}\|\Delta\beta_\theta\|(\varepsilon+\eta) + 2\delta(\varepsilon+\eta)$$

$$\le C_R|x_0-y_0|[1 + 6(R+1)e^{-\gamma\tau_0}\frac{|x_0-y_0|}{\theta^2} + 2\delta] +$$

$$+ \omega_{H,3(R+1)e^{-\gamma T}\|D\beta_\theta\|+1}(4\delta) + 6(R+1)e^{-\gamma\tau_0}\frac{\varepsilon+\eta}{\theta^2} + 2\delta(\varepsilon+\eta)$$

$$\le C_R(\bar{L}+2\delta)\theta^2[1 + (\bar{L}+2\delta) + 2\delta] + \omega_{H,3(R+1)e^{-\gamma T}\|D\beta_\theta\|+1}(4\delta)$$

$$+ 6(R+1)e^{-\gamma\tau_0}(\sqrt{\varepsilon} + \sqrt{\eta}) + 2\delta(\varepsilon+\eta)$$

where above we used the fact that

$$3(R+1)e^{-\gamma\tau_0}|D\beta_\theta(x_0-y_0)| \le \bar{L} + 2\delta \quad .$$

Letting $\delta \to 0$ implies

$$n'(\hat{\tau}) + \gamma m(\hat{\tau}) \le 2(C_R \bar{L}(1+\bar{L}) + 3(R+1)e^{-\gamma\hat{\tau}})(\sqrt{\varepsilon} + \sqrt{\eta})$$

and thus (2.12) with

$$K_1 = 2(C_R \bar{L}(1+\bar{L}) + 3(R+1)e^{-\gamma T}) \quad .$$

Remark 2.2. Note that the above proof gives a sharper estimate on $\|u_\varepsilon-u_\eta\|$ and thus $\|u_\varepsilon-u\|$, than the one stated in Proposition 2.1. In particular we proved that for $(x,\tau) \in \bar{Q}_T$

(2.21) $$|u_\varepsilon(x,\tau) - u_\eta(x,\tau)| \le e^{-\gamma\tau}(2(\bar{L})^2 + 2(C_R \bar{L}(1+\bar{L}) + 3(R+1))\tau)(\sqrt{\varepsilon} + \sqrt{\eta})$$

as one can easily check using Proposition 1.1 and the last inequality in the proof.

Proof of Lemma 2.1. Here we prove a more general estimate which has (2.4) and (2.5) as special cases. In particular, for $\varepsilon > 0$, let $H,\bar{H} \in C_b^2([0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfy (H2), (H3) and (H5) with the same constants C, C_R and $\gamma = \gamma_R \le 0$ for every $R > 0$. Moreover, let u_0, $\bar{u}_0 \in C_b^2(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$. If u_ε, $\bar{u}_\varepsilon \in C_b^{2,1}(\bar{Q}_T)$ are solutions of

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \varepsilon\Delta u_\varepsilon + H(\tau,x,u_\varepsilon,Du_\varepsilon) = 0 & \text{in } Q_T \\ u_\varepsilon(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial \bar{u}_\varepsilon}{\partial t} - \varepsilon\Delta \bar{u}_\varepsilon + \bar{H}(\tau,x,\bar{u}_\varepsilon,D\bar{u}_\varepsilon) = 0 & \text{in } Q_T \\ \bar{u}_\varepsilon(x,0) = \bar{u}_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

respectively, then for every $\tau \in [0,T]$

$$\sup_x |u_\varepsilon(x,\tau) - \bar{u}_\varepsilon(x,\tau)| \le e^{-\gamma\tau} \sup_x |u_0(x) - \bar{u}_0(x)| +$$

$$(2.22) \qquad + \tau e^{-\gamma\tau} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le\min(\|\bar{u}_\varepsilon\|,\|u_\varepsilon\|) \\ |p|\le\min(\bar{L}_\varepsilon, L_\varepsilon)}} |H(t,x,r,p) - \bar{H}(t,x,r,p)|$$

where $\qquad L_\varepsilon = \sup_{0\le\tau\le T} \|Du_\varepsilon(\cdot,\tau)\|, \ \bar{L}_\varepsilon = \sup_{0\le\tau\le T} \|D\bar{u}_\varepsilon(\cdot,\tau)\|$.

As usual, without any loss of generality, here we only prove that for every $\tau \in [0,T]$

$$\sup_x (u_\varepsilon(x,\tau) - \bar{u}_\varepsilon(x,\tau))^+ \le e^{-\gamma\tau} \sup_x (u_0(x) - \bar{u}_0(x))^+ +$$

$$(2.23) \qquad + \tau e^{-\gamma\tau} \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le\min(\|u_\varepsilon\|,\|\bar{u}_\varepsilon\|) \\ |p|\le\min(\bar{L}_\varepsilon, L_\varepsilon)}} |H(t,x,r,p) - \bar{H}(t,x,r,p)| \ .$$

To this end, let $m : [0,T] \to \mathbb{R}$ be defined by

$$m(\tau) = \sup_x (u_\varepsilon(x,\tau) - \bar{u}_\varepsilon(x,\tau))^+ \ .$$

We claim that m , which is a continuous function, is a viscosity solution of

$$m'(\tau) + \gamma m(\tau) \le \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le\min(\|u_\varepsilon\|,\|\bar{u}_\varepsilon\|) \\ |p|\le\min(L_\varepsilon,\bar{L}_\varepsilon)}} |H(t,x,r,p) - \bar{H}(t,x,r,p)| \ .$$

This, in view of Proposition 1.2 and Remark 1.3, proves (2.23). To prove the claim let $n \in C^\infty((0,T))$ and assume that $\hat{\tau} \in (0,T)$ is a strict maximum of $m - n$ on $I = [\hat{\tau}-\alpha, \hat{\tau}+\alpha]$ $(0,T)$ for some $\alpha > 0$. We want to show that

$$(2.24) \qquad n'(\hat{\tau}) + \gamma m(\hat{\tau}) \le \sup_{\substack{(x,t)\in\bar{Q}_T \\ |r|\le\min(\|u_\varepsilon\|,\|\bar{u}_\varepsilon\|) \\ |p|\le\min(L_\varepsilon,\bar{L}_\varepsilon)}} |H(t,x,r,p) - \bar{H}(t,x,r,p)| \ .$$

If $m(\hat{\tau}) = 0$, then $\hat{\tau}$ is minimum of n on I, therefore $n'(\hat{\tau}) = 0$ and

(2.24) is satisfied. So without any loss of generality we may assume that

(2.25) $$m(\hat{\tau}) > 0 \quad .$$

In this case let $\Phi : \mathbf{R}^N \times I \to \mathbf{R}$ be defined by

$$\Phi(x,\tau) = (u_\varepsilon(x,\tau) - \bar{u}_\varepsilon(x,\tau))^+ - n(\tau) \quad .$$

Since Φ is bounded on $\mathbf{R}^N \times I$, for every $\delta > 0$ there is a point $(x_1,\tau_1) \in \mathbf{R}^N \times I$ such that

$$\Phi(x_1,\tau_1) > \sup_{(x,\tau)\in\mathbf{R}^N\times I} \Phi(x,\tau) - \delta \quad .$$

Next we choose $\zeta \in C_0^\infty(\mathbf{R}^N)$ so that $0 \leq \zeta \leq 1$, $\zeta(x_1) = 1$, $|D\zeta| \leq 1$ and $|\Delta\zeta| \leq 1$ and define $\Psi : \mathbf{R}^N \times I \to \mathbf{R}^N$ by

$$\Psi(x,\tau) = \Phi(x,\tau) + 2\delta\zeta(x) \quad .$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,\tau_1) > \sup_{(x,\tau)\in\mathbf{R}^N\times I} \Phi(x,\tau) + \delta$$

there is a point $(x_0,\tau_0) \in \mathbf{R}^N \times I$ such that

(2.26) $$\Psi(x_0,\tau_0) \geq \Psi(x,\tau) \quad \text{for every} \quad (x,\tau) \in \mathbf{R}^N \times I \quad .$$

Moreover

(2.27) $$\begin{cases} \text{As} \quad \delta \downarrow 0 \quad \tau_0 \to \hat{\tau} \quad \text{and} \\ (u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0))^+ = u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0) \to m(\hat{\tau}) \quad . \end{cases}$$

Indeed suppose that as $\delta \downarrow 0$ $\tau_0 \to \bar{\tau} \in I$ along a subsequence (which for simplicity is denoted in the same way as the sequence). Then (2.26) implies

$$m(\tau_0) + 2\delta - n(\tau_0) \geq (u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0))^+ + 2\delta - n(\tau_0) \geq m(\tau) - n(\tau)$$

therefore as $\delta \downarrow 0$

$$m(\bar{\tau}) - n(\bar{\tau}) \geq m(\tau) - n(\tau) \quad \text{for every} \quad \tau \in I$$

which in view of the definition of $\hat{\tau}$ gives

$$\bar{\tau} = \hat{\tau} \quad .$$

In this case (2.26) also implies

$$m(\hat{\tau}) - n(\hat{\tau}) \geq \overline{\lim_{\delta\downarrow 0}} (u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0))^+ - n(\hat{\tau}) \geq$$

$$\geq \underline{\lim_{\delta\downarrow 0}} (u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0))^+ - n(\hat{\tau}) \geq m(\hat{\tau}) - n(\hat{\tau})$$

thus

$$\lim_{\delta\downarrow 0} (u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0))^+ = m(\hat{\tau}) \quad .$$

Finally, if along some subsequence $\delta\downarrow 0$

$$(u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0))^+ = 0$$

then $m(\hat{\tau}) = 0$ which contradicts (2.25).

Next observe that for δ sufficiently small τ_0 is an interior maximum point of $\tau \to u_\varepsilon(x_0,\tau) - \bar{u}_\varepsilon(x_0,\tau) - n(\tau)$ in I. Moreover x_0 is a maximum point of $x \to u_\varepsilon(x,\tau_0) - \bar{u}_\varepsilon(x,\tau_0) + 2\delta\zeta(x)$ in $\mathbb{R}^N$. The above, together with the fact that u_ε, $\bar{u}_\varepsilon$ satisfy the equations stated at the beginning of this proof, imply

$$n'(\tau_0) \leq 2\delta\varepsilon + \bar{H}(\tau_0,x_0,\bar{u}_\varepsilon(x_0,\tau_0), D\bar{u}_\varepsilon(x_0,\tau_0)) -$$

$$- H(\tau_0,x_0,u_\varepsilon(x_0,\tau_0), Du_\varepsilon(x_0,\tau_0)) \quad .$$

If (without any loss of generality) we assume that $\|u_\varepsilon\| = \min(\|u_\varepsilon\|, \|\bar{u}_\varepsilon\|)$ and $\bar{L}_\varepsilon = \min(L_\varepsilon,\bar{L}_\varepsilon)$, then

$$n'(\tau_0) + \gamma(u_\varepsilon(x_0,\tau_0) - \bar{u}_\varepsilon(x_0,\tau_0))^+ \leq 2\delta\varepsilon + \bar{H}(\tau_0,x_0,u_\varepsilon(x_0,\tau_0), D\bar{u}_\varepsilon(x_0,\tau_0)) -$$

$$- H(\tau_0,x_0,u_\varepsilon(x_0,\tau_0), D\bar{u}_\varepsilon(x_0,\tau_0) - 2\delta D\zeta(x_0)) \leq$$

$$\leq 2\delta\varepsilon + \omega_{H,\max(\|u_\varepsilon\|,\bar{L}_\varepsilon)}(2\delta) + \sup_{\substack{(x,t)\in\bar{\Omega}_T \\ |r|\leq\min\|u_\varepsilon\| \\ |p|\leq\bar{L}_\varepsilon}} |H(t,x,r,p) - \bar{H}(t,x,r,p)| \quad .$$

Letting $\delta\downarrow 0$, in view of (2.27), we obtain (2.24).

Since (2.4), (2.5) and (2.6) follow from (2.22) the same way that (1.19), (1.21) and (1.22) follow from (1.7) we omit their proof.

Remark 2.3. Estimates similar to (2.4) and (2.6) already exist in [6], where they are proved via arguments of the parabolic theory.

Now we continue with the proof of theorem 1. First however we give a short description of the arguments we are going to use. In particular, we approximate H and u_0 in a suitable way so that the resulting problems have viscosity solutions (by Proposition 2.1), which in view of Proposition 1.5 satisfy some estimates. Then using Proposition 1.4 we can conclude that (0.2) has a viscosity solution.

Proof of theorem 1. For the given u_0 and H and regardless of whether H satisfies (H4) or (H5) let $R_0 > 0$ and $T_0 > 0$ be such that

$$\begin{cases} 2\|u_0\| + C + 1 < R_0 \\ e^{-\gamma_{R_0} T_0}(\|u_0\| + (C+1)T_0) < R_0 \end{cases} \tag{2.28}$$

where C and γ_{R_0} are given by (H2) and (H3) respectively. Note that throughout the proof we assume that $\gamma_{R_0} \leq 0$. This does not impose any restrictions since one can always reduce the problem to this case.

The claim is that (0.2) has a unique viscosity solution on $\bar{Q}_{T_0}$. The uniqueness follows from Theorem 1.1 so here we have to establish the existence. To this end, we first observe that it suffices to assume that $u_0 \in C_b^2(\mathbb{R}^N)$. Indeed for the given $u_0 \in BUC(\mathbb{R}^N)$, we can find a sequence $u_{0,n} \in C_b^2(\mathbb{R}^N)$ so that

$$\|u_{0,n}\| \leq \|u_0\|$$

and

$$\|u_{0,n} - u_0\| \to 0 \quad \text{as} \quad n \to \infty .$$

If (0.2) has a viscosity solution $u_n \in BUC(\bar{Q}_{T_0})$ for every $u_{0,n} \in C_b^2(\mathbb{R}^N)$, then in view of (1.19)

$$\|u_n\| \le R_0$$

for every n, therefore by theorem 1.1

$$\|u_n - u_m\| \le e^{-\gamma T_0}\|u_{o,n} - u_{0,m}\|$$

i.e. there exists a $u \in BUC(\bar{Q}_{T_0})$ such that $u_n \to u$ uniformly on $\bar{Q}_{T_0}$ as $n \to \infty$. Then Proposition 1.3 implies that u is a viscosity solution of (0.2).

Next for every positive integer ℓ, let $\bar{H}_\ell : [0,T_0] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\text{(2.29)} \qquad \bar{H}_\ell(t,x,u,p) = W(P/\ell) \begin{cases} H(t,x,u,p) & \text{for } |u| \le R_0 \\ H(t,x,\frac{u}{|u|}R_0,p) & \text{for } |u| > R_0 \end{cases}$$

where $w \in C_0^\infty(\mathbb{R}^N)$ is such that

$$\text{(2.30)} \qquad \begin{cases} 0 \le w \le 1 \\ w(p) = 1 \quad \text{for} \quad |p| \le 1 \\ w(p) = 0 \quad \text{for} \quad |p| > 2 \ . \end{cases}$$

It is easy to see that for every ℓ

(i) $\bar{H}_\ell \in BUC([0,T_0] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$

(ii) $\sup_{(x,t)\in\bar{Q}_{T_0}} |\bar{H}_\ell(t,x,0,0)| = C$

(iii) $\bar{H}_\ell(t,x,r,p) - \bar{H}_\ell(t,x,s,p) \ge \gamma_{R_0}(r-s)$ for every $(x,t) \in \bar{Q}_{T_0}$

$p \in \mathbb{R}^N$ and $r \ge s$

(iv) $\bar{H}_\ell$ satisfies (H4) or (H5) depending on whether H satisfies (H4) or (H5) respectively. Also $\Lambda_R^{\bar{H}_\ell} \le \Lambda_R$ for $R > 0$ and $C_R^{H_\ell} \le C_{R_0}$ for $R > 0$.

Moreover observe that as $\ell \to \infty$, $\bar{H}_\ell(t,x,u,p) \to H(t,x,u,p)$ uniformly on $[0,T_0] \times \mathbf{R}^N \times [-R_0,R_0] \times B_N(0,R)$ for every $R > 0$.

Now for every ℓ, let $H_\ell \in C_b^2([0,T_0] \times \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N)$ be such that

(i) $\|H_\ell - \bar{H}_\ell\| < \frac{1}{\ell}$

(ii) $\sup_{(x,t)\in\bar{Q}_{T_0}} |H_\ell(t,x,0,0)| < C + 1$

(iii) $H_\ell(t,x,r,p) - H_\ell(t,x,s,p) \geq \gamma_{R_0}(r-s)$ for $(x,t) \in \bar{Q}_{T_0}$, $p \in \mathbf{R}^N$ and $r \geq s$

(iv) If H satisfies (H4), then H_ℓ also does and $\Lambda_R^{H_\ell}(\alpha) \leq \Lambda_{R+1}(\alpha)$ for $R > 0$

(v) If H satisfies (H5), then H_ℓ also does and $C_R^{H_\ell} \leq 2C_{R_0+1}$ for $R > 0$

(vi) Regardless of whether H satisfies (H4) or (H5), H_ℓ always satisfies (H5) for some constant $\bar{C}_R^\ell$ for $R > 0$.

Because of all the above properties of H_ℓ in view of Proposition 2.1, for every ℓ the problem

$$\begin{cases} \dfrac{\partial u_\ell}{\partial t} + H_\ell(t,x,u_\ell,Du_\ell) = 0 & \text{in } Q_{T_0} \\ u_\ell(x,0) = u_0(x) & \text{in } \mathbf{R}^N \end{cases}$$

has a unique viscosity solution $u_\ell \in BUC(\bar{Q}_{T_0})$. Moreover, because of the properties of H_ℓ and Proposition 1.5, for every $\tau \in [0,T_0]$ we have

$$\text{(2.31)} \qquad \begin{cases} \|u_\ell(\cdot,\tau)\| \leq e^{-\gamma_{R_0}\tau}(\|u_0\| + (C+1)\tau) < R_0 \\ \text{and} \\ \omega_{u_0(\cdot,\tau)}(\varepsilon) \leq f(\varepsilon) \quad \text{for} \quad \varepsilon < 1 \end{cases}$$

where $f : [0,\infty) \to [0,\infty)$ is so that $f(0^+) = 0$. In particular, if H satisfies (H4), then

$$f(\varepsilon) = e^{-\gamma_{R_0} T_0} \left(2\omega_{u_0}(\varepsilon) + T_0 \Lambda_{12R_0 e^{-\gamma_{R_0} T_0}+3}(2\varepsilon)\right) \tag{2.32}$$

and, if H satisfies (H5), then

$$f(\varepsilon) \le [e^{T_0(4C_{R_0+1}e^{-\gamma_{R_0} T_0} - \gamma_{R_0})}(\|Du_0\| + 2T_0 C_{R_0+1})]\varepsilon = \bar{\bar{L}}\varepsilon \; . \tag{2.33}$$

We want to show that $\{u_\ell\}$ is a Cauchy sequence in $BUC(\mathbb{R}^N)$ i.e. we want to show that for every $\alpha > 0$ there is a $\ell_0 = \ell_0(\alpha) > 0$ so that, if $\ell, \ell' > \ell_0$, then

$$\|u_\ell - u_{\ell'}\| \le \sup_{0 \le \tau \le T_0} \|u_\ell(\cdot,\tau) - u_{\ell'}(\cdot,\tau)\| < \alpha \; .$$

This, in view of Proposition 1.3, will finish the proof of the theorem. To this end and for arbitrary but fixed $\alpha > 0$, let $1 > \varepsilon > 0$ be so that

$$e^{-\gamma_{R_0} T_0} \omega_{u_0}(\varepsilon) < \alpha/3 \tag{2.34}$$

and

$$T_0 e^{-\gamma_{R_0} T_0} \Lambda_{12R_0 e^{-\gamma_{R_0} T_0}+3}(2\varepsilon) < \alpha/3 \tag{2.35}$$

if H satisfies (H4), or

$$2T_0 e^{-\gamma_{R_0} T_0} C_{R_0+1}(1+\bar{\bar{L}})\varepsilon < \alpha/3 \tag{2.36}$$

if H satisfies (H5). Having chosen ε as above next select ℓ_0 so that for $\ell, \ell' > \ell_0$

$$(2.37)\quad T_0 e^{-\gamma_{R_0} T_0} \sup_{\substack{(x,t)\in \bar{Q}_{T_0} \\ |r|\le R_0 \\ |p|\le \min\left(\frac{6R_0 e^{-\gamma_{R_0} T_0}}{\varepsilon} + 1,\ \bar{\bar{L}}\right)}} |H_\ell(t,x,r,p) - H_{\ell'}(t,x,r,p)| < \alpha/3$$

where if H does not satisfy (H5) $\bar{\bar{L}} = \infty$. Then in view of Proposition 1.4, we have that for $\tau \in [0,T_0]$ and $\ell,\ell' > \ell_0$

$$\|u_\ell(\cdot,\tau) - u_{\ell'}(\cdot,\tau)\| < \alpha$$

and thus the result.

Finally note that, if γ_R in (H2) is independent of R, we do not have to impose the restriction (2.28) on T and therefore we have existence for every $T > 0$.

Remark 2.4. In the case that γ_R is not independent of R we can not expect global time existence, as we can easily see from the simple ordinary differential equation

$$\begin{cases} u_t + u^2 = 0 \\ u(0) = c < 0 \end{cases} .$$

As a corollary of the above proof and Proposition 1.5, we have the following proposition which we state without proof.

Proposition 2.2. If H satisfies (H1), (H2), (H3) and (H5) and $u_0 \in C_b^{0,1}(\mathbb{R}^N)$, then (0.2) has a unique viscosity solution $u \in C_b^{0,1}(\bar{Q}_T)$.

Remark 2.5. A Lipschitz type condition in x is necessary in order to have solution Lipschitz in x. In particular, if $H \in BUC(\mathbb{R})$ is such that

$H(x) = x^{1/3}$ for $x \in [-1,1]$, then $u(x,t) = -tH(x) + 1$ is the viscosity solution of the problem

$$\begin{cases} u_t + H(x) = 0 \\ u(x,0) = 1 \end{cases}$$

but $u(\cdot,\tau) \in C_b^{0,1}(\mathbb{R})$ for $\tau \in [0,T]$.

Remark 2.6. Assumptions (H4) and (H5) are different. In particular, if H is independent of (t,u,p), then (H4) implies that H is uniformly continuous in x and (H5) that H is Lipschitz continuous in x. Moreover, there are functions which satisfy (H4) but not (H5) and vice versa. Indeed if $g : \mathbb{R} \to \mathbb{R}$ is Hölder continuous with exponent α then

$$H(x,p) = g(x)|p|^{\alpha-\varepsilon}$$

for $0 < \varepsilon < \alpha$ satisfies (H4) but not (H5). But if $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, then

$$H(x,p) = g(x)p$$

satisfies (H5) but not (H4).

Remark 2.7. One can prove Theorem 1 in the case that H satisfies (H5), using compactness arguments, once Proposition 1.5 is proved. However here we gave a constructive argument, which establishes the uniform convergence of solutions of approximate equations.

Section 3

We begin this section with the definition of the viscosity solution of (0.3). We have

Definition 3.1 ([1],[2]). Let $H \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ $\lambda > 0$ and $n \in C(\mathbb{R}^N)$. A function $u \in C(\mathbb{R}^N)$ is a viscosity solution of

$$u + \lambda H(x,u,Du) = n \quad \text{in} \quad \mathbb{R}^N$$

if for every $\phi \in C^\infty(\mathbb{R}^N)$

(3.1) if $u - \phi$ attains a local maximum at $x_0 \in \mathbb{R}^N$, then

$$u(x_0) + \lambda H(x_0,u(x_0),D\phi(x_0)) \leq n(x_0)$$

and

(3.2) if $u - \phi$ attains a local minimum at $x_0 \in \mathbb{R}^N$, then

$$u(x_0) + \lambda H(x_0,u(x_0),D\phi(x_0)) \geq n(x_0) \quad .$$

Next we state the theorem about the uniqueness of the viscosity solution of (0.3) as well as some other important *results of* [2] *concerning* this solution.

Theorem 3.1 (III.1[2]). Let $u,v \in BUC(\mathbb{R}^N)$ be viscosity solutions of the problems

$$u + \lambda H(x,u,Du) = n \quad \text{in} \quad \mathbb{R}^N \quad \text{and} \quad v + \lambda H(x,v,Dv) = m \quad \text{in} \quad \mathbb{R}^N$$

respectively, where $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies (H1), (H3) and either (H4) or (H5) and $n,m \in BUC(\mathbb{R}^N)$. Let $R_0 = \max(\|u\|,\|v\|)$ and $\gamma = \gamma_{R_0}$. Then

$$(1+\lambda\gamma)\|u-v\| \leq \|n-m\| \quad . \tag{3.3}$$

In particular, if $1 + \lambda\gamma > 0$, then (0.3) has a unique viscosity solution.

Proposition 3.1 (IV.1[2]). For $\varepsilon > 0$ let $u_\varepsilon \in C^2(\mathbb{R}^N)$ be a solution of

$$-\varepsilon\Delta u_\varepsilon + u_\varepsilon + \lambda H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) = v_\varepsilon \quad \text{in} \quad \mathbb{R}^N \quad .$$

Assume $H_\varepsilon \to H$ uniformly on $\mathbb{R}^N \times [-R,R] \times B_N(0,R)$ for each $R > 0$ and $v_\varepsilon \to v$ uniformly on $\mathbb{R}^N$. If $\varepsilon_n \downarrow 0$ and $u_{\varepsilon_n} \to u$ locally uniformly on $\mathbb{R}^N$,

then $u \in C(\mathbf{R}^N)$ is a viscosity solution of

$$u + \lambda H(x,u,Du) = v \quad \text{in} \quad \mathbf{R}^N \quad .$$

<u>Proposition 3.2 (I.2[2])</u>. Let $u_n \in C(\mathbf{R}^N)$ be a viscosity solution of $u_n + \lambda H_n(x,u_n,Du_n) = v_n$ in $\mathbf{R}^N$. Assume $H_n \to H$ uniformly on $\mathbf{R}^N \times [-R,R] \times B_n(0,R)$ for each $R > 0$ and $v_n \to v$ uniformly on $\mathbf{R}^N$. If $u_n \to u$ locally uniformly on $\mathbf{R}^N$, then $u \in C(\mathbf{R}^N)$ is a viscosity solution of

$$u + \lambda H(x,u,Du) = v \quad \text{in} \quad \mathbf{R}^N \quad .$$

Now we give a result which estimates the difference of the viscosity solutions of the two problems of the form (0.3). This estimate will be used later in order to derive several properties of the viscosity solution. To this end choose $\beta \in C_0^\infty(\mathbf{R}^N)$ as in (1.5). We have

<u>Proposition 3.3</u>. Let $u, \bar{u} \in BUC(\mathbf{R}^N)$ be viscosity solutions of the problems

$$u + \lambda H(x,u,Du) = v \quad \text{in} \quad \mathbf{R}^N \quad \text{and} \quad \bar{u} + \lambda \bar{H}(x,\bar{u},D\bar{u}) = \bar{v} \quad \text{in} \quad \mathbf{R}^N$$

respectively, where $H, \bar{H} : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ satisfy (H1) and (H3) with the same constant γ_R for each $R > 0$ and $v, \bar{v} \in BUC(\mathbf{R}^N)$. Let $R_0 = \max(\|u\|, \|\bar{u}\|)$ and $\gamma = \gamma_{R_0}$. If for $R > R_C$ and $\varepsilon > 0$, D_ε, A_ε are so that

$$D_\varepsilon = \{(x,y) \in \mathbf{R}^N \times \mathbf{R}^N : |x-y| < \varepsilon\}$$

and

$$A_\varepsilon = \{(x,y,r,p) \in R^N \times R^N \times R \times R^N : (x,y) \in D_\varepsilon,$$
$$|r| < \min(\|u\|, \|\bar{u}\|), \ |p| < \min(\frac{6R}{\varepsilon} + 1, L)\}$$

where

$$L = \min(\|Du\|, \|D\bar{u}\|)$$

and moreover

$$1 + \lambda\gamma > 0$$

then

$$\sup_{(x,y)\in D_\varepsilon} \{|u(x)-\bar{u}(y)| + 3R\beta_\varepsilon(x-y)\} \le \frac{1}{1+\lambda\gamma} \sup_{(x,y)\in D_\varepsilon} \{|v(x)-\bar{v}(y)| +$$

(3.4)

$$+ 3R(1+\lambda\gamma)\beta_\varepsilon(x-y)\} + \frac{\lambda}{1+\lambda\gamma} \sup_{(x,y,s,p)\in A_\varepsilon} |H(x,s,p) - \bar{H}(y,s,p)|$$

where $\beta_\varepsilon(\cdot) = \beta(\frac{\cdot}{\varepsilon})$.

<u>Remark 3.1</u>. The assumption that H, $\bar{H}$ satisfy (H3) with the same constant is not important. It is made only for simplicity.

<u>Proof of proposition 3.3</u>. It is obvious that (3.4) follows from

$$\sup_{(x,y)\in D_\varepsilon} \{(u(x)-\bar{u}(y))^{\pm} + 3R\beta_\varepsilon(x-y)\} \le \frac{1}{1+\lambda\gamma} \sup_{(x,y)\in D_\varepsilon} \{|v(x)-\bar{v}(y)| +$$

$(3.5)^{\pm}$

$$+ 3R(1+\lambda\gamma)\beta_\varepsilon(x-y)\} + \frac{\lambda}{1+\lambda\gamma} \sup_{(x,y,s,p)\in A_\varepsilon} |H(x,s,p)-\bar{H}(x,s,p)| \quad .$$

Here we prove only $(3.5)^+$ since $(3.5)^-$ follows exactly the same way. To this end observe that, if

$$\sup_{(x,y)\in D_\varepsilon} \{(u(x) - \bar{u}(y))^+ + 3R\beta_\varepsilon(x-y)\} \le 3R$$

then there is nothing to show. So we may assume

$$\sup_{(x,y)\in D_\varepsilon} \{(u(x) - \bar{u}(y))^+ + 3R\beta_\varepsilon(x-y)\} > 3R \quad . \tag{3.6}$$

In this case let $\Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\Phi(x,y) = (u(x) - \bar{u}(y))^+ + 3R\beta_\varepsilon(x-y) \quad .$$

Since Φ is bounded, for every $\delta > 0$ there is a point $(x_1,y_1) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

$$\Phi(x_1,y_1) > \sup_{(x,y)\in \mathbb{R}^N\times\mathbb{R}^N} \Phi(x,y) - \delta \quad .$$

Next select $\zeta \in C_0^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying $0 \le \zeta \le 1$, $\zeta(x_1,y_1) = 1$ and

$|D\zeta| \leq 1$ and define $\Psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi(x,y) = \Phi(x,y) + 2\delta\zeta(x,y) \quad .$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1) = \Phi(x_1,y_1) + 2\delta > \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} \Phi(x,y) + \delta$$

there exists a $(x_0,y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

(3.7) $\qquad \Psi(x_0,y_0) \geq \Psi(x,y)$ for every $(x_0,y_0) \in \mathbb{R}^N \times \mathbb{R}^N$.

We assert the following about (x_0,y_0)

$$(3.8)\quad \left\{\begin{array}{l} \text{If } \delta < R/2 \text{ then } |x_0-y_0| \leq \varepsilon \\ \text{and} \\ \text{as } \delta \downarrow 0,\ (u(x_0)-\bar{u}(y_0))^+ + 3R\beta_\varepsilon(x_0-y_0) = u(x_0)-\bar{u}(y_0) + 3R\beta_\varepsilon(x_0-y_0) \to \\ \qquad\qquad \sup_{(x,y)\in D_\varepsilon} \{(u(x)-\bar{u}(y))^+ + 3R\beta_\varepsilon(x-y)|\} \quad . \end{array}\right.$$

Indeed if $|x_0-y_0| > \varepsilon$, then in view of (3.7) and the definition of β_ε we have

$$2R + 2\delta \geq \Psi(x_0,y_0) \geq \Psi(x,y) \geq 3R$$

i.e. $\qquad \delta \geq R/2 \quad .$

Moreover (3.7) implies that

$$(u(x_0)-\bar{u}(y_0))^+ + 3R\beta_\varepsilon(x_0-y_0) + 2\delta \geq \sup_{(x,y)\in D_\varepsilon} \{(u(x)-\bar{u}(y))^+ + 3R\beta_\varepsilon(x-y)\} \quad .$$

So as $\delta \downarrow 0$

$$(u(x_0)-\bar{u}(y_0))^+ + 3R\beta_\varepsilon(x_0-y_0) \to \sup_{(x,y)\in D_\varepsilon} \{(u(x)-\bar{u}(y))^+ + 3R\beta_\varepsilon(x-y)\} \quad .$$

Finally observe that, if along some subsequence $\delta \downarrow 0$ it is $(u(x_0)-\bar{u}(y_0))^+ = 0$, then, in view of the above, we have

$$\sup_{(x,y)\in D_\varepsilon} \{(u(x)-\bar{u}(y))^+ + 3R\beta_\varepsilon(x-y)\} \leq 3R$$

which contradicts (3.6).

Now observe that x_0 is maximum point of $x \to u(x) + 3R\beta_\varepsilon(x-y_0) +$ $+ 2\delta\zeta(x,y_0)$ in $\mathbf{R}^N$ and y_0 is a minimum point of $y \to \bar{u}(y) -$ $- 3R\beta_\varepsilon(x_0-y) - 2\delta\zeta(x_0,y)$ in $\mathbf{R}^N$. In view of (3.1) and (3.2) we have

$$u(x_0) + \lambda H(x_0,u(x_0), -3RD\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) \leq v(x_0)$$

and

$$\bar{v}(y_0) \leq \bar{u}(y_0) + \lambda\bar{H}(y_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) + 2\delta D_y\zeta(x_0-y_0)) \quad .$$

Combining these two inequalities we obtain

$$u(x_0) - \bar{u}(y_0) \leq v(x_0) - \bar{v}(y_0) + \lambda\bar{H}(y_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) +$$

$$+ 2\delta D_y\zeta(x_0,y_0)) - \lambda H(x_0,u(x_0), -3RD\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) \quad .$$

To continue we assume that $\|\bar{u}\| = \min(\|u\|,\|\bar{u}\|)$. (If not one has to modify the rest of the proof in an obvious way.) Then in view of (H3) and (3.8) for δ sufficiently small we have

$$(1+\lambda\gamma)(u(x_0)-\bar{u}(y_0))^+ \leq v(x_0) - \bar{v}(y_0) + \lambda\bar{H}(y_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) +$$

$$+ 2\delta D_y\zeta(x_0,y_0)) - \lambda H(x_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0))$$

and, since $|x_0-y_0| \leq \varepsilon$ for $\delta < R/2$,

$$(1+\lambda\gamma)\{(u(x_0)-\bar{u}(y_0))^+ + 3R\beta_\varepsilon(x_0-y_0)\} \leq \sup_{(x,y)\in D_\varepsilon} \{|v(x)-\bar{v}(y)| +$$

$$+ 3R(1+\lambda\gamma)\beta_\varepsilon(x-y))\} + \lambda\bar{H}(y_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0)) -$$

$$- \lambda H(x_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) \quad .$$

Next observe that for $\delta < 1/2$

$$|-3RD\beta_\varepsilon(x_0-y_0) + 2\delta D_y\zeta(x_0,y_0)|, \ |-3RD\beta_\varepsilon(x_0-y_0) - 2\delta D_y\zeta(x_0,y_0)| \leq \frac{6R}{\varepsilon} + 1 \quad .$$

Moreover, if $L < \infty$ (note that the case $L = \infty$ is trivial) and (without any loss of generality) $L = \|Du\|$, then since x_0 is a maximum point of the mapping $x \to u(x) + 3R\beta_\varepsilon(x-y_0) + 2\delta\zeta(x,y_0)$ for $x \in \mathbf{R}^N$ we have

$$3R\beta_\varepsilon(x-y_0) + 2\delta\zeta(x,y_0) - 3R\beta_\varepsilon(x_0-y_0) - 2\delta\zeta(x_0,y_0) \le L|x-x_0| \quad .$$

But this implies that

$$|3RD\beta_\varepsilon(x_0-y_0) + 2\delta D_x\zeta(x_0,y_0)| \le L \quad .$$

Combining all the above we obtain

$$(1+\lambda\gamma)(u(x_0)-\bar{u}(y_0))^+ + 3R\beta_\varepsilon(x_0-y_0)) \le \sup_{(x,y)\in D_\varepsilon} \{|v(x)-\bar{v}(y)| + 3R(1+\lambda\gamma)\beta_\varepsilon(x-y)\} +$$

$$+ \lambda\bar{H}(y_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) -$$

$$- \lambda H(x_0,\bar{u}(y_0), -3RD\beta_\varepsilon(x_0-y_0) - 2\delta D_x\zeta(x_0,y_0)) +$$

$$+ \lambda\omega_{\bar{H},\max(\frac{6R}{\varepsilon}+1,\ R)}(4\delta)$$

therefore

$$(1+\lambda\gamma)\{(u(x_0)-\bar{u}(y_0))^+ + 3R\beta_\varepsilon(x_0-y_0)\} \le \sup_{(x,y)\in D_\varepsilon} \{|v(x)-\bar{v}(y)| +$$

$$+ 3R(1+\lambda\gamma)\beta_\varepsilon(x-y)\} + \lambda \sup_{(x,y,s,p)A_\varepsilon} |H(x,s,p) - \bar{H}(y,s,p)| + \lambda\omega_{\bar{H},\max(\frac{6R}{\varepsilon}+1,R)}(4\delta).$$

Letting $\delta \downarrow 0$ in the last inequality we get $(3.5)^+$.

Next we use Proposition 3.3 to establish several properties of the viscosity solution $u \in BUC(\mathbb{R}^N)$ of (0.3). In particular the next proposition gives a priori bounds for the norm, the modulus of continuity and the Lipschitz constant of u. Moreover, it gives an estimate for $\|u-v\|$, if $v \in C_b^{0,1}(\mathbb{R}^N)$.

<u>Proposition 3.4</u>. Let $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (H1) and (H3). If for $v \in BUC(\mathbb{R}^N)$, $u \in BUC(\mathbb{R}^N)$ is a viscosity solution of (0.3), let $R > \|u\|$ and $\gamma = \gamma_R$. If $1 + \lambda\gamma > 0$, the following are true

(a) If H satisfies (H2) then

$$\|u\| \le \frac{1}{1+\lambda\gamma}(\lambda C + \|v\|) \tag{3.9}$$

where C is given by (H2).

(b) If H satisfies (H4) then for $1 > r > 0$

$$\omega_u(r) \le \frac{1}{1+\lambda\gamma}(2\omega_v(r) + \lambda\Lambda_{12R+2}(2r)) \quad . \tag{3.10}$$

(c) If H satisfies (H5) and $u, v \in C_b^{0,1}(\mathbb{R}^N)$, then

(3.11) $$\|Du\| \le \frac{1}{1+\lambda\gamma} [\|Dv\| + \lambda C_R(1 + \|Du\|)]$$

where C_R is given by (H5). Moreover if $1 + \lambda(\gamma - C_R) > 0$ then

(3.12) $$\|Du\| \le \frac{1}{1+\lambda(\gamma-C_R)} (\|Dv\| + \lambda C_R) .$$

(d) If H satisfies (H2) and $v \in C_b^{0,1}(\mathbb{R}^N)$ then

(3.13) $$\|u-v\| \le \frac{\lambda}{1+\lambda\gamma} \sup_{\substack{x\in\mathbb{R}^N \\ |r|\le\|u\| \\ |p|\le\|Dv\|}} |H(x,r,p)| .$$

<u>Proof</u>. (a) We apply Proposition 3.3 to u and $\bar{u} = 0$, which is a viscosity solution of the problem

$$0 + \lambda 0 = 0 \quad \text{in} \quad \mathbb{R}^N .$$

Then for $\varepsilon > 0$, (3.4) implies

$$\|u\| + 3R \le \sup_{(x,y)\in D_\varepsilon} \{|u(x)| + 3R\beta_\varepsilon(x-y)\} \le \frac{1}{1+\lambda\gamma} \sup_{(x,y)\in D_\varepsilon} |v(x)| + 3R +$$

$$+ \frac{\lambda}{1+\lambda\gamma} \sup_{(x,y,r,p)\in A_\varepsilon} |H(x,r,p)| .$$

But in this case

$$A_\varepsilon = \{(x,y,r,p) : |x-y| \le \varepsilon,\ |r| \le \min(\|u\|,0),\ |p| \le \min(\tfrac{6R}{\varepsilon} + 1,0)\}$$

$$= \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x-y| \le \varepsilon\} .$$

So

$$\|u\| \le \frac{1}{1+\lambda\gamma} (\|v\| + \lambda C) .$$

(b) For r fixed $(1 > r > 0)$ let $\xi \in \mathbb{R}^N$ be such that

$$|\xi| \le r .$$

If $\bar{u} : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\bar{u}(x) = u(x+\xi)$$

then $\bar{u} \in BUC(\mathbb{R}^N)$. Moreover $\bar{u}$ is a viscosity solution of

$$\bar{u} + \lambda H(x + \xi, \bar{u}, D\bar{u}) = v(x+\xi) \quad \text{in} \quad \mathbb{R}^N .$$

To see this observe that, if for $\phi \in C^\infty(\mathbb{R}^N)$, x_0 is a local maximum of $\bar{u} - \phi$, then $x_0 + \xi$ is a local maximum of $u - \psi$ where $\psi(y) = \phi(y-\xi)$. By (3.1) we have

$$u(x_0+\xi) + \lambda H(x_0+\xi,\ u(x_0+\xi),\ D\psi(x_0+\xi)) \leq v(x_0+\xi)$$

therefore

$$\bar{u}(x_0) + \lambda H(x_0+\xi,\ \bar{u}(x_0),\ D\phi(x_0)) \leq v(x_0+\xi) \quad .$$

Similarly one can check the case $\bar{u} - \phi$ has a local minimum at x_0.

Now applying Proposition 3.3 to u, $\bar{u}$ for $\varepsilon = r$ we have

$$\sup_x |u(x)-u(x+\xi)| + 3R \leq \sup_{(x,y)\in D_r} \{|u(x)-u(y+\xi)| + 3R\beta_r(x-y)\} \leq$$

$$\leq \frac{1}{1+\lambda\gamma} \sup_{(x,y)\in D_r} |v(x)-v(y+\xi)| + \frac{1}{1+\lambda\gamma} 3R(1+\lambda\gamma) +$$

$$+ \frac{\lambda}{1+\lambda\gamma} \sup_{(x,y,s,p)\in A_r} |H(x,s,p) - H(y+\xi,s,p)| \quad .$$

But in view of (H4)

$$\sup_{(x,y,s,p)\in A_r} |H(x,s,p) - H(y+\xi,s,p)| \leq \sup_{\substack{|x-y|\leq r \\ |s|\leq \|u\| \\ |p|\leq \frac{6R}{r}+1}} |H(x,s,p) - H(y+\xi,s,p)| \leq$$

$$\leq \Lambda_{12R+2}(2R) \quad .$$

Moreover

$$\sup_{(x,y)\in D_r} |v(x) - v(y+\xi)| \leq 2\omega_v(r)$$

thus the result.

(c) For $\xi \in \mathbb{R}^N$ define $\bar{u} : \mathbb{R}^N \to \mathbb{R}$ by

$$\bar{u}(x,\tau) = u(x+\xi,\tau) \quad .$$

Then $\bar{u} \in C_b^{0,1}(\mathbb{R}^N)$ is a viscosity solution of

$$\bar{u} + H(x+\xi,\bar{u},D\bar{u}) = v(x+\xi) \quad \text{in} \quad \mathbb{R}^N .$$

Applying Proposition 3.3 to u, $\bar{u}$ for $\varepsilon > 0$ we have:

$$\sup_x |u(x)-u(x+\xi)| + 3R \le \frac{1}{1+\lambda\gamma} \sup_{(x,y)\in D_\varepsilon} |v(x)-v(y+\xi)| + 3R +$$

$$+ \lambda \sup_{\substack{(x,y)\in D_\varepsilon \\ |s|\le\|u\| \\ |p|\le\|Du\|}} |H(x,s,p) - H(y+\xi,s,p)|$$

therefore

$$\sup_x |u(x)-u(x+\xi)| \le \frac{1}{1+\lambda\gamma} (\|Dv\| + \lambda C_R(1 + \|Du\|))(\varepsilon+|\xi|) .$$

Letting $\varepsilon \downarrow 0$ we obtain (3.11). If $1 + \lambda(\gamma-C_R) > 0$, (3.12) follows from (3.11).

(d) Applying Proposition 3.3 to u and $v \in C_b^{0,1}(\mathbb{R}^N)$, which is a viscosity solution of

$$v + \lambda 0 = v \quad \text{in} \quad \mathbb{R}^N$$

for $\varepsilon > 0$ we have

$$\|u-v\| + 3R \le \frac{1}{1+\lambda\gamma} \sup_{(x,y)\in D_\varepsilon} |v(x)-v(y)| + 3R + \frac{\lambda}{1+\lambda\gamma} \sup_{\substack{|x-y|\le\varepsilon \\ |s|\le\|u\| \\ |p|\le\|Dv\|}} |H(x,s,p)| .$$

Letting $\varepsilon \to 0$ we obtain (3.13).

Remark 3.2. In the case that H is independent of x, one can deduce (3.10), (3.11) and (3.12) directly from (3.3) ([2]).

Section 4

We begin this section with a result concerning the existence of the viscosity solution of (0.3), in the case that H and v are sufficiently smooth functions. In particular, we show that the solution of the viscosity approximation

$$(4.1)_\varepsilon \qquad -\varepsilon\Delta u_\varepsilon + u_\varepsilon + \lambda H(x,u_\varepsilon,Du_\varepsilon) = v \quad \text{in } \mathbb{R}^N$$

converges as $\varepsilon \to 0$ uniformly on $\mathbb{R}^N$ to a function $u \in BUC(\mathbb{R}^N)$, which is then, by Proposition 3.1, the viscosity solution (0.3). Moreover we give an explicit estimate on $\|u-u_\varepsilon\|$.

<u>Proposition 4.1</u>. Let $H \in C_b^2(\mathbb{R}^N\times R\times\mathbb{R}^N)$ satisfy (H2), (H3) (with $\gamma = \gamma_R$ for $R > 0$) and (H5). For $\lambda > 0$ so that $1 + 2\lambda\gamma > 0$, $1 + \lambda(\gamma-1) > 0$ and $1 + \lambda(\gamma-C_R) > 0$, where $R > 2\|v\| + C$ and C, C_R are given by (H2), (H5), $\varepsilon > 0$ and $v \in C_b^2(\mathbb{R}^N)$, let $u_\varepsilon \in C^2(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$ be the solution of $(4.1)_\varepsilon$. Then there exists $u \in BUC(\mathbb{R}^N)$ such that $u_\varepsilon \to u$ uniformly on $\mathbb{R}^N$ as $\varepsilon \to 0$. u is the viscosity solution of (0.3) in $\mathbb{R}^N$ and moreover

$$(4.2) \qquad \|u-u_\varepsilon\| \leq K\sqrt{\varepsilon}$$

where K is a constant which depends only on $\|v\|$ and $\|Dv\|$.

<u>Proof</u>. The existence of such an u_ε follows from standard theory (see in particular [7]). Moreover it is also known that, under our assumptions on H, v, $u_\varepsilon \in C_b^{0,1}(\mathbb{R}^N)$. In order to show the existence of u it suffices to show that as $\varepsilon \to 0$ $\{u_\varepsilon\}$ forms a Cauchy family in $BUC(\mathbb{R}^N)$. Indeed then there exists $u \in BUC(\mathbb{R}^N)$ such that $u_\varepsilon \to u$ uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0$. By Proposition 3.1 and Theorem 3.1 u is the viscosity solution of (0.3). To this end we show that there exists a constant K, which depends only on $\|Dv\|$ and $\|v\|$, such that for ε, $\eta > 0$

(4.3)
$$\|u_\varepsilon - u_\eta\| \le K(\sqrt{\varepsilon} + \sqrt{\eta}) \quad .$$

This also will prove (4.2), if we let $\eta \to 0$. To prove (4.3) we need the following lemma.

Lemma 4.1. If H, v, ε, λ, R and u_ε are as in Proposition 4.1, then

(4.4)
$$\|u_\varepsilon\| \le \frac{1}{1+\lambda\gamma}(\|v\| + \lambda C)$$

where C is given by (H2) and

(4.5)
$$\|Du_\varepsilon\| \le \frac{1}{1+\lambda(\gamma - C_R)}(\|Dv\| + \lambda C_R) = \bar{L}$$

where C_R is given by (H5).

We first complete the proof of the proposition and then prove the lemma. Observe that it suffices to show that there exists a constant K, which depends only on $\|v\|$, $\|Dv\|$, such that for $\varepsilon, \eta > 0$

$(4.6)^{\pm}$
$$\|(u_\varepsilon - u_\eta)^{\pm}\| \le K(\sqrt{\varepsilon} + \sqrt{\eta}) \quad .$$

Here we establish only $(4.6)^+$, since $(4.6)^-$ can be proved in exactly the same way. To this end observe that, if $\|(u_\varepsilon - u_\eta)^+\| = 0$, there is nothing to prove. So we may assume that

(4.7)
$$\|(u_\varepsilon - u_\eta)^+\| > 0 \quad .$$

In this case and for $\theta = \sqrt[4]{\varepsilon} + \sqrt[4]{\eta}$ let $\Phi : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ be defined by

$$\Phi(x,y) = (u_\varepsilon(x) - u_\eta(y))^+ + 3(R+1)\beta_\theta(x-y)$$

where R is as in the statement of the proposition and $\beta_\theta(\cdot) = \beta(\frac{\cdot}{\theta})$ with β given by (2.9). Since Φ is bounded, for every $\delta > 0$ there is a point (x_1, y_1) in $\mathbf{R}^N \times \mathbf{R}^N$ such that

$$\Phi(x_1,y_1) > \sup_{(x,y)\in \mathbf{R}^N \times \mathbf{R}^N} \Phi(x,y) - \delta \quad .$$

Next select $\zeta \in C_0^\infty(\mathbf{R}^N)$ so that $0 \le \zeta \le 1$, $\zeta(x_1,y_1) = 1$, $|D\zeta| \le 1$ and $|\Delta\zeta| \le 1$ and define $\Psi : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ by

$$\Psi(x,y) = \Phi(x,y) + 2\delta\zeta(x,y) \quad .$$

Since $\Psi = \Phi$ off the support of ζ and

$$\Psi(x_1,y_1) = \Phi(x_1,y_1) + 2\delta > \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} \Phi(x,y) + \delta$$

there is a point $(x_0,y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

(4.8) $$\Psi(x_0,y_0) \geq \Psi(x,y) \quad \text{for every} \quad (x,y) \in \mathbb{R}^N \quad .$$

We claim that (x_0,y_0) has the following properties

(4.9) $$\begin{cases} \text{For} \quad \delta < \min\left(\dfrac{\|(u_\varepsilon-u_\eta)^+\|}{2}, \dfrac{1}{24}\right) \\ |x_0,y_0| \leq \theta, \ |x_0-y_0| \leq (\bar{L}+2\delta)\theta^2, \ (u_\varepsilon(x_0)-u_\eta(y_0))^+ > 0 \quad \text{and} \\ (u_\varepsilon(x_0)-u_\eta(y_0))^+ > \|(u_\varepsilon-u_\eta)^+\| - 2\delta \end{cases}$$

where $\bar{L}$ is given by (4.5). Indeed, since in view of (4.4) $\|u_\varepsilon\| \leq R$, if $|x_0-y_0| > \theta$, (4.8) implies

$$2(R+1) + 2\delta \geq \Psi(x_0,y_0) \geq \Psi(x,x) \geq 3(R+1)$$

which contradicts the fact that $\delta < 1/24$. Moreover, for every $x \in \mathbb{R}^N$, it is

$$(u_\varepsilon(x_0)-u_\eta(y_0))^+ + 3(R+1) + 2\delta \geq \Psi(x_0,y_0) \geq \Psi(x,x) \geq (u_\varepsilon(x)-u_\eta(x))^+ + + 3(R+1)$$

therefore

$$(u_\varepsilon(x_0)-u_\eta(y_0))^+ \geq \|(u_\varepsilon-u_\eta)^+\| - 2\delta$$

and by the choice of δ

$$(u_\varepsilon(x_0)-u_\eta(y_0))^+ = u_\varepsilon(x_0)-u_\eta(y_0) \quad .$$

In this case and since $\|Du_\varepsilon\| \leq \bar{L}$ we have

$$u_\varepsilon(x_0)-u_\eta(y_0) + 3(R+1)\beta_\theta(x_0-y_0) + 2\delta \geq u_\varepsilon(y_0)-u_\eta(y_0) + 3(R+1)$$

therefore

$$3(R+1)\beta_\theta(x_0-y_0) \geq 3(R+1) - 2(R+1) - 2\delta(3(R+1))$$

which implies

$$\beta_\theta(x_0-y_0) > \frac{1}{4} \quad .$$

But then, in view of (2.9), it is

$$\beta_\theta(x_0-y_0) = 1 - \frac{|x_0-y_0|^2}{\theta^2} \quad . \tag{4.10}$$

Moreover, (4.8) also implies that, for δ sufficiently small, x_0 is a maximum point of the mapping $x \to u_\varepsilon(x) + 3(R+1)\beta_\theta(x-y_0) + 2\delta\zeta(x,y_0)$, therefore for $x \in \mathbb{R}^N$

$$3(R+1)\beta_\theta(x-y_0) + 2\delta\zeta(x,y_0) - 3(R+1)\beta_\theta(x_0-y_0) - 2\delta\zeta(x_0,y_0) \le u_\varepsilon(x_0)-u_\varepsilon(x) \le$$
$$\le \bar{L}|x-x_0| \le 6(R+1)\bar{L}|x-x_0| \quad .$$

This gives

$$|3(R+1)D\beta_\theta(x_0-y_0) + 2\delta D_x\zeta(x_0,y_0)| \le \bar{L} \le 6(R+1)\bar{L} \tag{4.11}$$

and by (4.10)

$$|x_0-y_0| \le (\bar{L}+2\delta)\theta^2 \quad .$$

Next observe that, in view of (4.8) and (4.9) x_0 is a maximum point of $x \to u_\varepsilon(x) + 3(R+1)\beta_\theta(x-y_0) + 2\delta\zeta(x,y_0)$ and y_0 is a minimum point of $y \to u_\eta(y) - 3(R+1)\beta_\theta(x_0-y) - 2\delta\zeta(x_0,y)$. This, together with the fact that u_ε, $u_\eta \in C^2(\mathbb{R}^N)$ are solutions of $(4.1)_\varepsilon$, $(4.1)_\eta$ respectively, implies that

$$u_\varepsilon(x_0)-u_\eta(y_0) \le -3(R+1)\beta_\theta(x_0-y_0)(\varepsilon+\eta) + \lambda H(y_0,u_\eta(y_0),-3(R+1)D\beta_\theta(x_0-y_0) +$$
$$+ 2\delta D_y\zeta(x_0,y_0)) - \lambda H(x_0,u_\varepsilon(y_0),-3(R+1)D\beta_\theta(x_0-y_0)-2\delta D_x\zeta(x_0,y_0)) + v(x_0) - v(y_0) \; .$$

But then using (4.9) and the properties of H, v and λ we have

$$(1+\lambda\gamma)(u_\varepsilon(x_0)-u_\eta(y_0))^+ \le \|Dv\|\,|x_0-y_0| + 3(R+1)\,\frac{\varepsilon+\eta}{\theta^2} +$$
$$+ \lambda\omega_{H,\max(R,\,\frac{3(R+1)}{\theta}\|D\beta\|+1)}(4\delta) + \lambda H(y_0,u_\varepsilon(y_0),-3(R+1)D\beta_\theta(x_0-y_0)-$$
$$-2\delta D_x\zeta(x_0,y_0)) - \lambda H(y_0,u_\varepsilon(y_0),-3(R+1)D\beta_\theta(x_0-y_0)-2\delta D_x\zeta(x_0,y_0))$$

therefore, since $\theta^2 \le 2(\sqrt{\varepsilon} + \sqrt{\eta})$ and $\frac{\varepsilon+\eta}{\theta^2} \le \sqrt{\varepsilon} + \sqrt{\eta}$, and by (4.11)

$$(1+\lambda\gamma)\|(u_\varepsilon-u_\eta)^+\| \le (1+\lambda\gamma)2\delta + 2\|Dv\|(\bar{L}+2\delta)(\sqrt{\varepsilon}+\sqrt{\eta}) +$$

$$+ \lambda\omega_{H,\max(R,\ \frac{3(R+1)}{\theta}\|D\beta\|+1)}(4\delta) + \lambda C_R(1+\bar{L})2(\bar{L}+2\delta)(\sqrt{\varepsilon}+\sqrt{\eta}) \quad .$$

Letting $\delta \downarrow 0$ we obtain

$$\|(u_\varepsilon-u_\eta)^+\| \le \frac{1}{1+\lambda\gamma}(2\|Dv\|\bar{L} + 2\lambda C_R(1+\bar{L})\bar{L})(\sqrt{\varepsilon}+\sqrt{\eta}) \tag{4.12}$$

and thus the result.

<u>Proof of Lemma 4.1</u>. Here we prove a more general estimate which has (4.4) and (4.5) as special cases. In particular, for $\varepsilon > 0$ let $H, \bar{H} \in C_b^2(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfy (H2), (H3) and (H5) with the same constants C, C_R and $\gamma = \gamma_R$ for $R > 0$. Moreover, let $v, \bar{v} \in C_b^2(\mathbb{R}^N)$ and choose $R_0 > 0$ so that

$$\max(2\|v\|+ C,\ 2\|\bar{v}\| + C) < R_0 \quad .$$

If $\lambda > 0$ is so that $1 + 2\lambda\gamma > 0$, $1 + \lambda(\gamma-1) > 0$ and $1 + \lambda(\gamma-C_{R_0}) > 0$ and u_ε, $\bar{u}_\varepsilon \in C^2(\mathbb{R}^N) \cap BUC(\mathbb{R}^N)$ are solutions of

$$-\varepsilon\Delta u_\varepsilon + u_\varepsilon + H(x,u_\varepsilon,Du_\varepsilon) = v \quad \text{and} \quad -\varepsilon\Delta\bar{u}_\varepsilon + \bar{u}_\varepsilon + H(x,\bar{u}_\varepsilon,D\bar{u}_\varepsilon) = \bar{v}$$

then

$$\|u_\varepsilon-\bar{u}_\varepsilon\| \le \frac{1}{1+\lambda\gamma}\Big(\|v-\bar{v}\| + \lambda \sup_{\substack{x\in\mathbb{R}^N \\ |r|\le\min(\|u_\varepsilon\|,\|\bar{u}_\varepsilon\|) \\ |p|\le\min(L_\varepsilon,\bar{L}_\varepsilon)}} |H(x,r,p) - \bar{H}(x,r,p)|\Big) \tag{4.13}$$

where $L_\varepsilon = \|Du_\varepsilon\|$ and $\bar{L}_\varepsilon = \|D\bar{u}_\varepsilon\|$.

As usual and without any loss of generality here we show only

$$\|(u_\varepsilon-\bar{u}_\varepsilon)^+\| \le \frac{1}{1+\lambda\gamma}\Big(\|v-\bar{v}\| + \lambda \sup_{\substack{x\in{}^N \\ |r|\le\min(\|u_\varepsilon\|,\|\bar{u}_\varepsilon\|) \\ |p|\le\min(L_\varepsilon,\bar{L}_\varepsilon)}} |H(x,r,p) - \bar{H}(x,r,p)|\Big) \quad . \tag{4.14}$$

To this end observe that, if $\|(u_\varepsilon-\bar{u}_\varepsilon)^+\| = 0$, there is nothing to prove. So we assume that

$$\|(u_\varepsilon-\bar{u}_\varepsilon)^+\| > 0 \quad .$$

In this case let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\Phi(x) = (u_\varepsilon(x)-\bar{u}_\varepsilon(x))^+ \quad .$$

Since Φ is bounded, for every $\delta > 0$ there is a point $x_1 \in \mathbb{R}^N$ such that

$$\Phi(x_1) > \sup_{x\in\mathbb{R}^N} \Phi(x) - \delta \quad .$$

Let $\zeta \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \zeta \le 1$, $\zeta(x_1) = 1$, $|D\zeta| \le 1$ and $|\Delta\zeta| \le 1$ and define $\Psi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi(x) = \Phi(x) + 2\delta\zeta(x) \quad .$$

Since $\Psi = \Phi$ off the support of δ and

$$\Psi(x_1) = \Phi(x_1) + 2\delta > \sup_{x\in\mathbb{R}^N} \Phi(x) + \delta$$

there is a point $x_0 \in \mathbb{R}^N$ such that

(4.15) $$\Psi(x_0) \ge \Psi(x) \text{ for every } x \in \mathbb{R}^N \quad .$$

Then for $\delta < \dfrac{\|(u_\varepsilon-\bar{u}_\varepsilon)^+\|}{2}$ it is easy to check that

$$(u_\varepsilon(x_0)-\bar{u}_\varepsilon(x_0))^+ = u_\varepsilon(x_0) - \bar{u}_\varepsilon(x_0) > \|(u_\varepsilon-\bar{u}_\varepsilon)^+\| - 2\delta \quad .$$

But then x_0 is a maximum point of $x \to u_\varepsilon(x) - \bar{u}_\varepsilon(x) + 2\delta\zeta(x)$. This, together with the fact that $u_\varepsilon, \bar{u}_\varepsilon \in C^2(\mathbb{R}^N)$ satisfy the equations stated at the beginning of the proof, implies

$$u_\varepsilon(x_0) - \bar{u}_\varepsilon(x_0) \le -2\delta\varepsilon + \lambda(\bar{H}(x_0,\bar{u}_\varepsilon(x_0),D\bar{u}_\varepsilon(x_0)) -$$

$$- H(x_0,u_\varepsilon(x_0),Du_\varepsilon(x_0) + 2\delta D\zeta(x_0)) \quad .$$

If we assume (without any loss of generality) that $\|u_\varepsilon\| = \min(\|u_\varepsilon\|,\|\bar{u}_\varepsilon\|)$ and $\bar{L}_\varepsilon = \min(L_\varepsilon,\bar{L}_\varepsilon)$, then

$$(1+\lambda\gamma)(u_\varepsilon(x_0)-\bar{u}_\varepsilon(x_0))^+ \le -2\delta\varepsilon + \lambda \sup_{\substack{x\in\mathbb{R}^N \\ |r|\le\|u_\varepsilon\| \\ |p|\le\bar{L}_\varepsilon}} |H(x,r,p) - \bar{H}(x,r,p)| +$$

$$+ \lambda\omega_{H,\max(\|u_\varepsilon\|,\bar{L}_\varepsilon)}(2\delta) \quad .$$

Letting $\delta \downarrow 0$ we obtain the result.

Since (4.4) and (4.5) follow from (4.13) the same way that (3.9), (3.12) follow from (3.4) we omit their proof.

Remark 4.1. Once one has (4.4) and (4.5) the existence of the viscosity solution $u \in BUC(\mathbb{R}^N)$ of (0.3) under the assumptions of Proposition 4.1 follows immediately from usual compactness arguments. The only reason we give a different proof is to establish the explicit estimate on $\|u_\varepsilon - u\|$.

Now we continue with the proof of Theorem 2. As in the case of Theorem 1 here we approximate H and u_0 in a suitable way so that the resulting problems have viscosity solutions (by Proposition 4.1). Using the a priori estimates we have about the viscosity solution together with Proposition 3.3, we can conclude that (0.3) has a solution.

Proof of theorem 2. For the given n and H and regardless of whether H satisfies (H4) or (H5) let $R_0 > 0$ be such that

(4.16) $$2\|n\| + C + 1 \le R_0$$

where C is given by (H2). Then choose $\lambda_0 > 0$ so that for $0 < \lambda < \lambda_0$

(4.17) $$\begin{cases} 1 + 2\lambda\gamma_{R_0} > 0 \\ 1 + \lambda(\gamma_{R_0} - 1) > 0 \end{cases}$$

and

(4.18)
$$1 + \lambda(\gamma_{R_0} - 2C_{R_0+1}) > 0$$

in the case that H satisfies (H5), where γ_{R_0} is given by (H3) and is assumed to be $\gamma_{R_0} < 0$ and C_{R_0+1} is given by (H5). The claim is that, for every λ such that $0 < \lambda < \lambda_0$, (0.3) has a unique viscosity solution. The uniqueness follows from Theorem 1.1 and the choice of λ since by Proposition 3.4(a), any solution $u \in BUC(\mathbb{R}^N)$ is such that

$$\|u\| < \frac{1}{1+\lambda\gamma}(\|n\|+\lambda C) < R_0 \quad .$$

Here we establish the existence. To this end we first observe that it suffices to assume $n \in C_b^2(\mathbb{R}^N)$. Indeed for the given $n \in BUC(\mathbb{R}^N)$ we can find a sequence $n_m \in C_b^2(\mathbb{R}^N)$ so that

$$\|n_m\| < \|n\|$$

and

$$\|n_m - n\| \to 0 \quad \text{as} \quad m \to \infty \quad .$$

If we know that (0.3) has a viscosity solution for $n \in C_b^2(\mathbb{R}^N)$, then for every n and λ as above

$$u_m + \lambda H(x,u_m,Du_m) = n_m$$

will have a viscosity solution $u_m \in BUC(\mathbb{R}^N)$ such that

$$\|u_m\| < R_0 \quad .$$

But then theorem 1.1 implies

$$\|u_m - u_\ell\| < \frac{1}{1+\lambda\gamma}\|n_m - n_\ell\|$$

i.e. there exists a $u \in BUC(\mathbb{R}^N)$ such that $u_m \to u$ uniformly on $\mathbb{R}^N$ as $m \to \infty$. Then by Proposition 3.2, u is the viscosity solution of (0.3).

Next for every positive integer ℓ let $\bar{H}_\ell : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\bar{H}_\ell(x,u,p) = w(p/\ell)\begin{cases} H(x,u,p) & \text{for } |u| < R_0 \\ H(x, \frac{u}{|u|}R_0, p) & \text{for } |u| > R_0 \end{cases}$$

where $w \in C_0^\infty(\mathbb{R}^N)$ is as in (2.30). It is easy to see that for every ℓ

(i) $\overline{H}_\ell \in BUC(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$

(ii) $\sup_x |\overline{H}_\ell(x,0,0)| = C$

(iii) $\overline{H}_\ell(x,r,p) - \overline{H}_\ell(x,s,p) \geq \gamma_{R_0}(r-s)$ for every $x \in \mathbb{R}^N$, $p \in \mathbb{R}^N$ and $r \geq s$

(iv) $\overline{H}_\ell$ satisfies (H4) or (H5) depending on whether H satisfies (H4) or (H5) respectively. Moreover $\Lambda_R^{\overline{H}_\ell} \leq \Lambda_R$ for $R > 0$ and $C_R^{H_\ell} \leq C_{R_0}$ for $R > 0$.

Also observe that as $\ell \to \infty$, $\overline{H}_\ell(x,u,p) \to H(x,u,p)$ uniformly on $\mathbb{R}^N \times [-R_0,R_0] \times B_N(0,R)$ for every $R > 0$.

Now for each ℓ let $H_\ell \in C_b^2(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ be such that

(i) $\|H_\ell - \overline{H}_\ell\| \leq \frac{1}{\ell}$

(ii) $\sup_x |H_\ell(x,0,0)| \leq C + 1$

(iii) $H_\ell(x,r,p) - H_\ell(x,s,p) \geq \gamma_{R_0}(r-s)$ for $x \in \mathbb{R}^N$, $p \in \mathbb{R}^N$ and $r \geq s$

(iv) If H satisfies (H4), then H_ℓ also does and $\Lambda_R^{H_\ell} \leq \Lambda_{R+1}$ for $R > 0$

(v) If H satisfies (H5), then H_ℓ also does and $C_R^{H_\ell} \leq 2C_{R_0+1}$ for $R > 0$

(vi) Regardless of whether H satisfies (H4) or (H5), H_ℓ always satisfies (H5) for some constant $\overline{C}_R^\ell$ for $R > 0$.

Because of all the above in view of Proposition 4.1, for each ℓ the problem

$$u_\ell + H_\ell(x,u_\ell,Du_\ell) = n \quad \text{in} \quad \mathbb{R}^N$$

has a unique viscosity solution $u_\ell \in BUC(\mathbb{R}^N)$. Moreover, because of (i) above, Proposition 3.4 and (4.17), (4.18), for every ℓ we have

$$
(4.19)\qquad \begin{cases} \|u_\ell\| \le \dfrac{1}{1+\lambda\gamma_{R_0}} (\|n\| + (C+1)\lambda) \le R_0 \\ \text{and} \\ \omega_{u_\ell}(\varepsilon) \le f(\varepsilon) \quad \text{for} \quad \varepsilon > 1 \end{cases}
$$

where $f : [0,\infty) \to [0,\infty)$ is such that $f(0^+) = 0$. In particular, if H satisfies (H4), then for $\varepsilon < 1$

$$
f(\varepsilon) = \frac{1}{1+\lambda\gamma_{R_0}} (2\omega_n(\varepsilon) + \lambda\Lambda_{12R_0+3}(2\varepsilon))
$$

and if H satisfies (H5), then

$$
f(\varepsilon) = \frac{1}{1+\lambda(\gamma_{R_0} - 2C_{R_0+1})} (\|Dn\| + 2C_{R_0+1}\lambda)\varepsilon = \bar{\bar{L}}\varepsilon .
$$

We want to show that $\{u_\ell\}$ is a Cauchy sequence in $BUC(\mathbb{R}^N)$ i.e. that for every $\alpha > 0$ there is a $\ell_0 = \ell_0(\alpha) > 0$ so that if $\ell,\ell' > \ell_0$, then

$$
\|u_\ell - u_{\ell'}\| < \alpha .
$$

This, in view of Proposition 3.2 will finish the proof of the theorem. To this end and for $\alpha > 0$ arbitrary but fixed let $1 > \varepsilon > 0$ be so that

$$
(4.20)\qquad \frac{1}{1+\lambda\gamma_{R_0}} \omega_n(\varepsilon) < \alpha/3
$$

and

$$
(4.21)\qquad \frac{\lambda}{1+\lambda\gamma_{R_0}} \Lambda_{12R_0+3}(2\varepsilon) < \alpha/3
$$

if H satisfies (H4), or

$$
(4.22)\qquad \frac{\lambda}{1+\lambda\gamma_{R_0}} 2C_{R_0+1}(1+\bar{\bar{L}})\varepsilon < \alpha/3
$$

if H satisfies (H5). Having chosen ε as above, next select ℓ_0 so that for $\ell,\ell' > \ell_0$

$$\frac{\lambda}{1+\lambda\gamma_{R_0}} \sup_{\substack{x\in\mathbb{R}^N \\ |r|<R_0 \\ |p|<\min(\frac{6R_0}{\epsilon}+1,\bar{\bar{L}})}} |H_\ell(x,r,p) - H_{\ell'}(x,r,p)| < \alpha/3 \tag{4.23}$$

where in the case that H does not satisfy (H5), $\bar{\bar{L}} = \infty$. Then, in view of Proposition 3.3, for $\ell,\ell' > \ell_0$ we have

$$\|u_\ell - u_{\ell'}\| < \alpha$$

and thus the result.

As a corollary of the above proof and Proposition 3.5, we state without a proof the following proposition.

<u>Proposition 4.2</u>. If H satisfies (h1), (H2), (H3) and (H5) and $n \in C_b^{0,1}(\mathbb{R}^N)$, then (0.3) has a unique viscosity solution $u \in C_b^{0,1}(\mathbb{R}^N)$.

<u>Remark 4.2</u>. If H is independent of u, then the above proof gives $\lambda_0 = \infty$. If H satisfies (H3) and either (H4) or (H5), so that the constants are independent of R, then λ_0 is independent of $\|n\|$.

<u>Remark 4.3</u>. One can prove theorem 2 by using compactness arguments, once Propositions 3.4 and 4.1 are proved. Here we gave a more constructive proof to establish the uniform convergence of solutions of approximate equations.

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Equations of Hamilton-Jacobi type arise in many areas of application, including the calculus of variations, control theory and differential games. Recently M. G. Crandall and P. L. Lions introduced the class of "viscosity" solutions of these equations and proved uniqueness within this class. This paper discusses the existence of these solutions under assumptions closely related to the ones which guarantee the uniqueness.

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